MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 97 No. 2

MEETING OF 1936 DECEMBER 11

Mr. J. H. Reynolds, President, in the Chair.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

M. C. L. Bhalla, M.A., Principal, University Tutorial Academy, 61 Brandreth Road, Lahore, India (proposed by P. S. Lall) ;

B. C. Browne, M.A., Pembroke College, Cambridge (proposed by Sir G. Lenox-Conyngham) ;

Joseph William Henry Crabb, 4 Gordon Place, London, W.C. 1 (proposed by C. C. L. Gregory) ;

Roy Kenneth Marshall, A.B., A.M., Ph.D., Wilson College Observatory, Chambersburg, Pennsylvania, U.S.A. (proposed by H. D. Curtis) ; and

Humphrey Montague Smith, High Bryn, 73 Repton Road, Orpington, Kent (proposed by H. Spencer Jones).

Seventy-three presents were announced as having been received since the last meeting, including :—

James Stokeley, Stars and Telescopes. With Preface by Dr. W. S. Adams.

Professor L. M. Milne-Thomson and Mr. E. A. Chamberlain were appointed auditors of the Treasurer's accounts for 1936.

Dr. E. P. Hubble gave an account of the progress made to date with the 200-inch telescope.

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AIR TEMPERATURE DURING THE TOTAL SOLAR ECLIPSE OF 1936 JUNE 19.

John L. Haughton, D.Sc., F.Inst.P.

It is often reported that a great fall in temperature takes place during a total eclipse of the Sun. The author has made measurements of the air temperature during two previous eclipses and has carried out further observations during the eclipse of 1936 June 19 on board the deck of the s.s. *Strathaird* in the neighbourhood of Lat. N. 37° 08′, Long. E. 23° 30′. In addition to observing air temperatures, some photographs were taken of the eclipse.

The author was assisted by Mrs. M. L. V. Haughton, D.Sc., and Mr. J. Cartland, M.C., M.Sc., who read the temperature and called out the times respectively.

The apparatus used for measuring the air temperature consisted of a multiple thermocouple made of ten pairs of iron constantan wires. The wires were soldered together in pairs at one end by means of a large drop of silver solder, which was then hammered out into a disc about 6 mm. in diameter and 0·3 mm. thick. The couples were connected in series by soldering, the soldered ends being insulated by shellac varnish, and were packed into a glass tube, which was then filled with paraffin wax. The tube was fitted into a "thermos" flask filled with crushed ice. The couples were bent at right angles where they left the tube, so that they lay in a horizontal plane about 25 cm. from the base of the flask, the junctions being in a circle of about 30 cm. radius. The E.M.F. generated by the couple was read on a potentiometer reading from 0 to 18 millivolts in steps of 1 millivolt, the out-of-balance current being indicated on a portable millivoltmeter which could be read to about 0·05 mv. Voltage for the potentiometer was obtained from a 1·5 v. dry cell, and in order to avoid having to use a standard cell, a resistance of about 6 ohms was included in the circuit, and the deflection of the millivoltmeter when placed across this resistance was observed when the instrument was balanced against a standard cell before leaving England. Before taking any readings the potentiometer was adjusted till this deflection was obtained, and the millivoltmeter was then transferred to the couple circuit. The couple was calibrated on returning to England, and it was found that, in the neighbourhood of the temperatures prevailing during the eclipse, 1 mv. = 2°·1 C., so that it was possible to read to about 0°·1 C. The instrument responded very rapidly to change of temperature—if the hand was placed within a few centimetres of *one* of the couples an immediate change of reading was observed.

On the *Strathaird* the instrument was mounted on the floor of the monkey

island and was screened from the wind and from direct sunshine by the walls of the island. The temperatures were read at 10-minute intervals from 0500 till totality, at 10, 30 and 50 seconds after totality and then at 10-minute intervals till 0640. Readings were taken three days before the eclipse and one day after in order to observe the way the temperature changed in the absence of the eclipse. The author wishes to thank Mr. Farrer for his assistance on two of these occasions. Of these four days, only one (that after the eclipse) was of any use, as the weather on the 16th, 17th and 18th was cloudy, with fog on the 18th and slight rain on the 16th, while the 19th and 20th were clear and bright. The results obtained on the 20th are plotted as a dotted line in fig. 1. At about 0650 the deck of the monkey island was washed and the instrument had to be moved. The marked fall in temperature will be noted, otherwise a steady rise occurred.

The full curve in fig. 1 gives the results obtained on the morning of the eclipse. It will be noticed that the temperature rose at first, possibly slightly faster than on the next day, but that after about 10 minutes the effect of the eclipse began to make itself felt and the temperature fell steadily till about the moment of totality, when it fell more rapidly till about 1 minute after totality; the temperature then rose rapidly for about 10 minutes, after which it became almost steady. At this point it was noted that a wind had risen, which may be the cause of the cessation of temperature rise.

The chain curve in fig. 1 represents the course which the time-temperature curve might be assumed to have taken had there been no eclipse. It starts from just below the highest temperature observed on the 19th, before the eclipse began to have any appreciable effect; and from there is drawn parallel to the curve obtained next day. The maximum fall from this curve—just after totality—is 1°·5 C. The smallness of this fall may be partly accounted for by the conditions of the eclipse: the low altitude of the Sun and the thermostating effect of the large mass of water, but it is obvious that the fall in air temperature will in any case be small and that the "great fall in temperature" referred to as occurring in other eclipses must be a fall in radiation temperature and not in air temperature. It may be of interest to compare these results with those of two previous eclipses, though the results are not strictly comparable.

The eclipse of 1927 June 29 took place, as far as the north of England was concerned, under similar conditions to the present one.* The altitude of the Sun at totality was 12°·5 as compared with 9° from the *Strathaird*, and both eclipses occurred in the early morning. On the other hand, the eclipse of 1932 August 31 occurred in Canada in the late afternoon ($16^h\ 30^m$ local time), and the altitude of the Sun was over 30°.† In the 1927 and 1936 eclipses the normal temperature was rising, in the 1932 eclipse it was falling. These differences make accurate comparison impossible, but the facts observed are given in the table on p. 99.

Unfortunately, comparative observations when there was no eclipse were not made in 1927 or 1932.

* *M.N.*, **87**, 692, 1927.
† *M.N.*, **93**, 9, 1932.

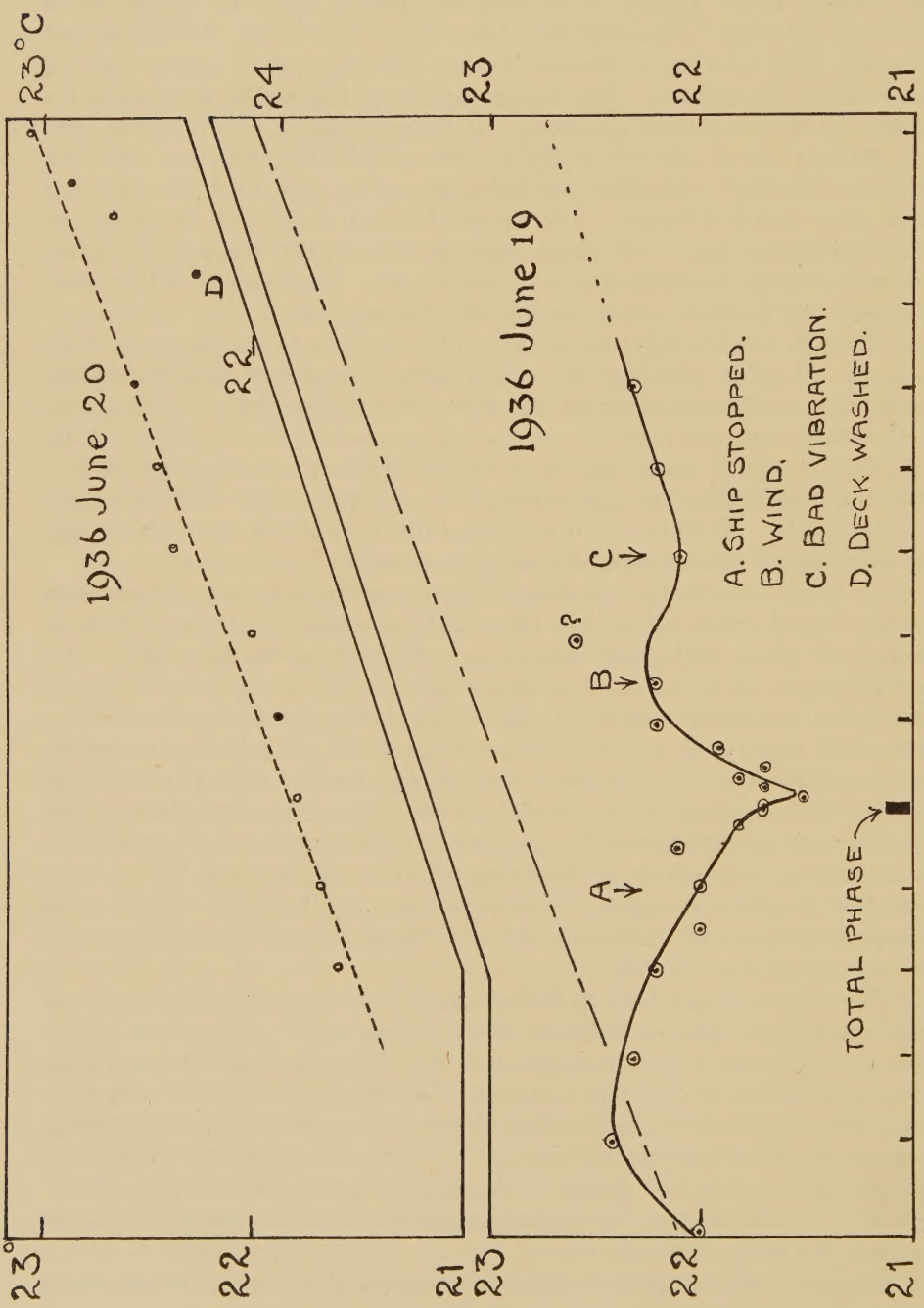

23°C
24
23
22
21
1936 June 20
1936 June 19
D
22
A
B
C
?
A. Ship stopped.
B. Wind.
C. Bad vibration.
D. Deck washed.
TOTAL PHASE
0500
20
40
0600
20
40
0700
23°
22
21
23
22
21

Fig. 1.

1	2	3	4	5
1927	0·0	0·0	5·0	0·0
1932	3·5	1·0	21·0	1·0 ?
1936	0·9	0·5	21·5	1·5

Column 1. Date of eclipse.
2. Total observed fall in temperature between 1st and 2nd contacts. (°C.)
3. Fall of temperature within a few minutes of totality. (°C.)
4. Lowest temperature reached (°C.)
5. Fall presumably due to the eclipse. (°C.)

General Remarks

The amount of illumination during totality was such that neither Mrs. Haughton nor Mr. Cartland experienced any difficulty in reading the temperature or the time respectively. Some of the prominences remained visible after the 3rd contact for a time that is estimated as 15 to 20 seconds.

Acknowledgments

The author wishes to thank the P. & O. Company, and in particular the officers of the s.s. *Strathaird*, for the facilities they afforded for carrying out the work. He is also grateful to Dr. Comrie for making all the arrangements which made it possible to get satisfactory temperature records, and to Mrs. Haughton, Mr. Cartland and Mr. Farrer for their assistance.

National Physical Laboratory,
Teddington.

TWO NEW SHORT-PERIOD SPECTRUM VARIABLES.

D. L. Edwards.

Short-period variables of the β Canis Majoris type have recently been receiving some attention at the Norman Lockyer Observatory from the point of view of line intensities in their spectra. The object of such work is to establish (when possible) the fact of variation in these stars, with an indication as to its general nature, followed by an attempt to test the agreement or otherwise of their periods with the periods of light variation or radial velocity. When sufficient data have accumulated it may also be possible to correlate the periods with spectral type or other characteristics of the stars. With this purpose in view the stars suggested by Henroteau * have been placed on the working list of the 12-inch McClean objective prism instrument, together with a number of other stars of similar spectral types in order to increase, if possible, the number of known objects of this class. The latter stars were included on account of discrepancies in the

* *Pub. Dom. Obs., Ottawa*, 9, 23, 1925.

published values of radial velocity or of spectral type, which in either case might be due to short-period variation. Two of these stars (ι Herculis and ψ Eridani) have already been found to show definite periodic variations of line intensities, and form the subject of this present note.

The instruments and methods of work have already been sufficiently described * in an account of similar variations in 42 Camelopardalis. The only difference in the present case lies in the fact that the lines at λλ 4388 (*He*) and 4481 (Mg^+) were the only ones measured by the wedge. It was shown previously that these lines varied in opposite directions (*i.e.* one increased in intensity when the other decreased); the difference of intensity should therefore be a more sensitive quantity to show the variations. The line 4388 appeared to show the effect much more strongly than 4471, and was therefore chosen, rather than the more adjacent helium line, as a comparison for 4481. It was thought that the possible loss of accuracy due to the comparison of lines at the slightly greater separation was more than compensated for by the gain derived from the greatly increased amplitude of variation.

ι *Herculis*

This star (H.D. 160762, position in 1900 R.A. $17^h\ 36^m{\cdot}6$, Dec. $+46°\ 4'$, mag. 3·79, spectral type B3) was placed on the observing list on account of inconsistencies in radial velocity measurements. Observations made by Henroteau † in 1920 suggested that the velocity had changed since 1901 and 1902. Also a series of observations on one night seemed to indicate a small range of radial velocity of less than 15 km./sec.; but the observations were inconclusive, though Henroteau remarked that "plotting on cross-section paper of the Ottawa velocities make us suspect a period of about $0^d{\cdot}4$." This variability, however, was not confirmed by later observations. Plaskett and Pearce ‡ remark that, "Although announced as binary by Henroteau . . . the three-prism observations of Lick and Yerkes are in excellent agreement with our determinations and indicate that the velocity is constant." Also in the Lick § catalogue of radial velocities is found the remark: "Henroteau's conclusion that the velocity is variable is not confirmed by the observations obtained at Victoria, Mount Wilson and Lick." It should, however, be noted that none of these observatories has made a lengthy series of observations during one night, and their results are therefore not inconsistent with a small variation of very short period.

No photometric observations have been found which suggest any magnitude variations in this star. If present at all they must be of very small amplitude.

Three series of photographs have been taken at Sidmouth, on the nights of 1936 August 21, 22 and 23. The plates used were Barnet "Super Press," and the exposures were all of seven minutes duration, with intervals of

* *M.N.*, **93**, 729, 1933.
† *Pub. Dom. Obs., Ottawa*, **5**, 345, 1922.
‡ *Pub. Dom. Ap. Obs., Vict.*, **5**, 50, 1931.
§ *Pub. Lick Obs.*, **18**, 209, 1932.

twenty minutes between the times of mid-exposure. The photographs in any one series were all taken on one plate. The existence of changes in the intensities of some of the spectral lines was at once evident, and appear to be of a similar nature to those already described for 42 Camelopardalis (*loc. cit.*), but are not quite so pronounced as in the latter star. As already remarked, the lines measured by the wedge method were 4481 and 4388, and by using the ratio of intensities of these lines the principal errors due to plate emulsion, exposure, development, etc. were eliminated or greatly reduced. In Table I are given the dates and times of exposure of the negatives, with the intensity ratios 4388/4481, as measured by the wedge, and the phase (calculated from the mean period and assumed date of maximum at J.D. 2428402·411).

TABLE I

Neg.	Date (1936)	J.D. 2428000 +	Phase	Wedge Ratio (4388/4481)
	d h m		d	
5110	Aug. 21 21 31·5	402·397	0·129	1·23
5111	51·5	·411	·000	1·34
5112	22 11·5	·425	·014	1·30
5113	31·5	·439	·028	1·26
5114	51·5	·452	·041	1·23
5115	23 11·5	·466	·055	1·23
5116	31·5	·480	·069	1·22
5117	51·5	·494	·083	1·17
5118	22 0 11·5	·508	·097	1·17
5119	31·5	·522	·111	1·23
5120	51·5	·536	·125	1·16
5121	1 11·5	·550	·139	1·26
5122	31·5	·564	·010	1·36
5123	21 8·5	403·381	·110	1·19
5124	28·5	·395	·124	1·18
5125	48·5	·409	·138	1·30
5126	22 8·5	·423	·008	1·35
5127	28·5	·436	·021	1·30
5128	48·5	·450	·035	1·27
5129	23 8·5	·464	·049	1·23
5130	28·5	·478	·063	1·20
5131	48·5	·492	·077	1·21
5132	23 0 8·5	·506	·091	1·16
5133	28·5	·520	·105	1·16
5134	48·5	·534	·119	1·23
5135	20 53·5	404·370	·095	1·19
5136	21 13·5	·384	·109	1·17
5137	33·5	·398	·123	1·22
5138	53·5	·412	·137	1·33
5139	22 13·5	·426	·007	1·33
5140	33·5	·440	·021	1·29
5141	53·5	·454	·035	1·24
5142	23 13·5	·468	·049	1·23

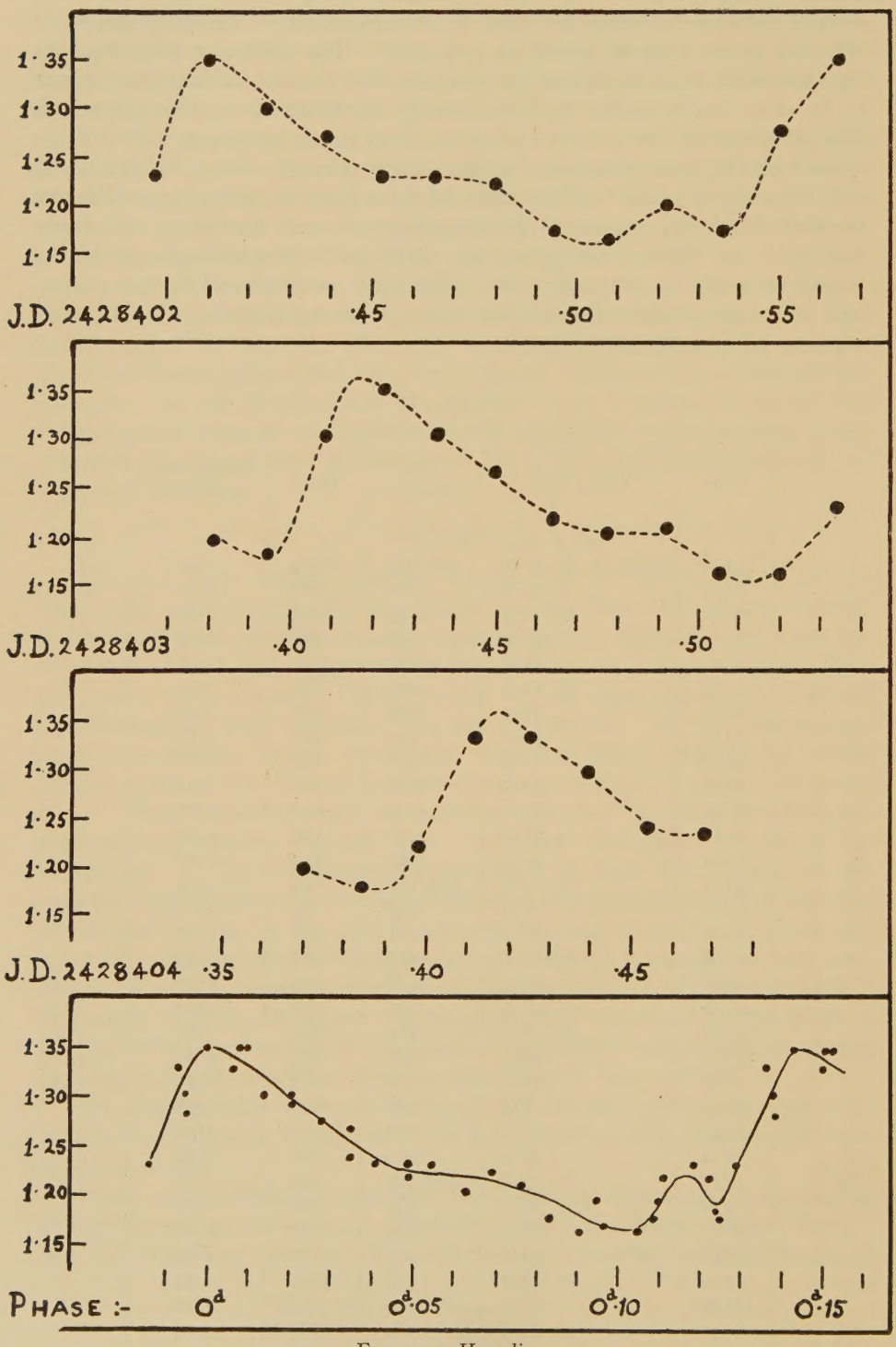


FIG. 1.—*ι Herculis.*

Variation of rates of wedge readings, 4388/4481 *on three dates, with resulting mean curve.*

Fig. 1 shows the variations of these wedge ratios on each of the three nights of observation, and also a mean curve obtained by plotting all the observed values against phase. An approximate value of the period from the curve of August 21 was sufficient to determine the number of periods between the first maximum (on August 21, $9^h\ 51^m.5$) and the last (August 23, $10^h\ 3^m.5$), giving a mean value of $P = 0^d.1434$ ($3^h\ 27^m$). The rise to maximum, as in the case of 42 Camelopardalis, is very much steeper than the fall to minimum, but there is a curious "hump" just at the beginning of the rise, which is fairly definitely indicated in each of the three curves. There is also a well-marked flattening in the downward part of the curve from about $0^d.04$ to $0^d.07$ after maximum. The shape of the curve is repeated almost identically (within the errors of observation) on each of the three nights, and the comparatively small amount of scatter of points round the mean curve suggests that both period and amplitude are probably constant over the complete series.

The following is a summary of the data deduced from the photographs :—

Period	$0^d.1434$ ($=3^h\ 27^m$).
Minimum to maximum . .	$0^d.04$ (approx.).
Maximum to minimum . .	$0^d.10$,,
Secondary maximum . .	$0^d.03$ before maximum.
Spectral type . . .	B3 to B4.

ψ Eridani

In a note * on some peculiarities in B-type stars a list was given in which the spectral types as determined at Sidmouth showed marked divergences from the H.D. types, and it was suggested that one possible reason for this was an actual change in the spectrum. With the idea of testing the possibility of periodic changes in spectral class, or of confirming the disagreement with Harvard, many of these stars have been put on the observing list with β Canis Majoris variables. The first star, ψ Eridani (H.D. 32249, position in 1900 R.A. $4^h\ 56^m.6$, Dec. $-7°\ 20'$, mag. 4·81, spectral type B8) was observed during the night of 1934 January 20, and showed definite changes in its spectrum analogous to those seen in 42 Cam. and ι Her. The changes, however, while easily measurable, are quite inadequate to account for the great difference between the Harvard classification (B8) and that of the Norman Lockyer Observatory (B3). Eleven photographs were taken on the one night, and in no case could the spectrum be classified as later than B3·5 or possibly B4.

The exposures given were all on one plate and were each of 10 minutes duration, with intervals of 20 minutes between the times of mid-exposure. The wedge-reading ratios of 4388 to 4481 were again measured, and the results are given in fig. 2. This curve is similar in general form to those of ι Her., including the pause in the descending portion ; but the maximum seems to be less sharply defined, and there are not sufficient observations

* *M.N.*, **92**, 393, 1932.

to show whether there is a small secondary maximum or not. The period appears to be about $3^h\ 20^m$ or possibly longer, which is of the same order as that of ι Herculis. The amplitude also appears to be practically the same

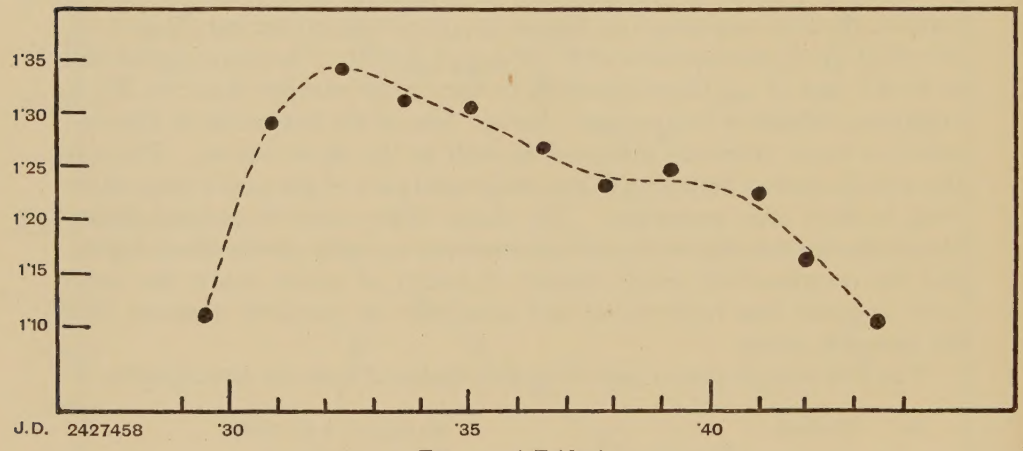


FIG. 2.—*ψ Eridani.*

Variation of rates of wedge readings, 4388/4481 on 1934 January 20.

as in the latter star, though the observations are insufficient to give good values of either of these quantities.

Unfortunately it has not yet been possible to take a fresh or more lengthy series of observations of this star, but the general character of the changes as shown in the diagram seems to suggest that it should definitely be classed amongst the short-period β Canis Majoris variables.

Summary

Variations of line intensities have been studied in the two stars ι Herculis and ψ Eridani.

In particular, the relative intensities of $\lambda\lambda$ 4388 and 4481 have been measured with a wedge, and the results show definite periodic variations in both stars.

In the case of ι Her. the period is found to be $0^d{\cdot}1434$ with a steep rise to maximum intensity difference ($0^d{\cdot}04$) and a slower decline ($0^d{\cdot}10$).

ψ Eridani shows a somewhat similar curve, and both stars appear definitely to belong to the class of short-period variables typified by β Canis Majoris.

The Norman Lockyer Observatory,
Sidmouth:
1936 *October* 28.

ON THE PERIODIC VARIATION IN THE COLOURS OF THE TWO EQUATORIAL BELTS OF JUPITER. (THIRD PAPER.)

A. Stanley Williams.

The first paper on this subject was published in the *Monthly Notices*, **59**, 376, 1899, and the second in **90**, 696, 1930, of the same publication. In the former paper a period of 12·08 years was derived from the observations then available, and in the latter all those up to the end of 1929 were considered, the period of variation being increased to 12·5 years. We now have available the observations of six additional years, but these are of great importance as they indicate a decided shortening of the period. Hence it seems desirable that they should be published at once without any delay. They show that the period of variation may now definitely be taken to be the same as that of the planet's revolution round the Sun (11·86 years), as had indeed been formerly surmised. Table I gives the additional observations in the same form as in Table I of the second paper. They are the result of the work of 125 additional nights' observations.

TABLE I

Observed Redness of the Belts

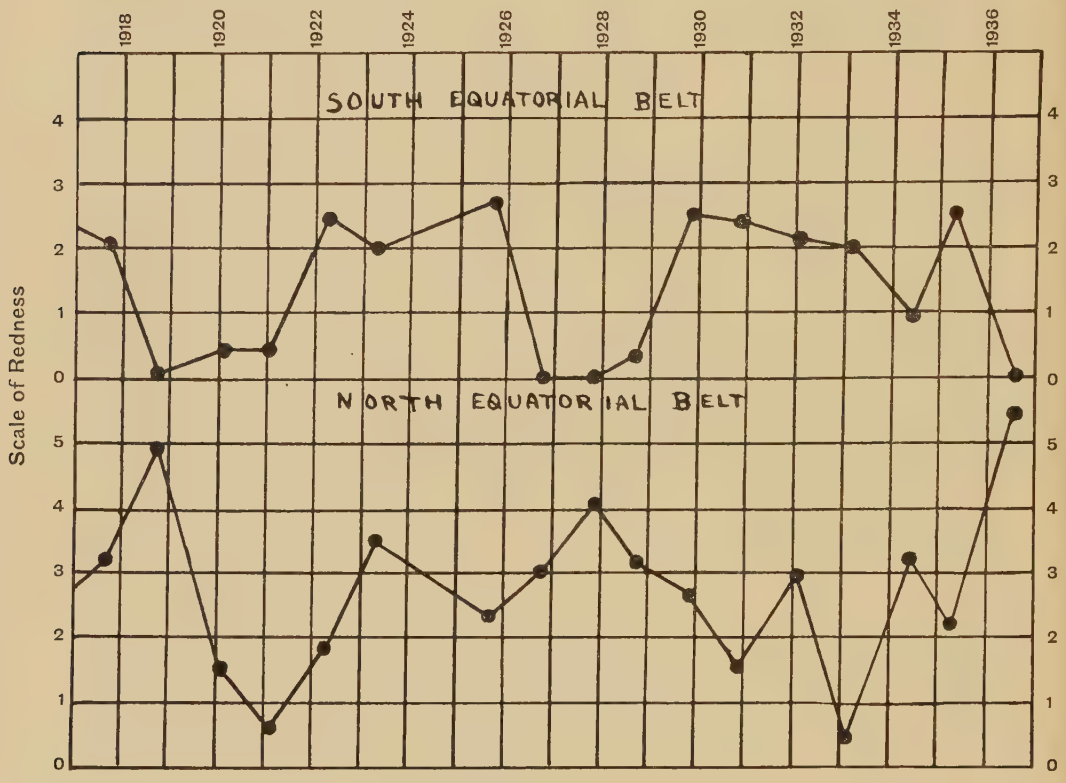
No.	Date	SEB.	NEB.	Nights	Remarks
32	1930·9	2·4	1·6	17	Little colour about the planet.
33	1932·1	2·2	2·9	26	Both belts rosy.
34	1933·2	2	0·4	16	Colour of SEB. difficult to make out.
35	1934·4	0·9	3·2	16	NEB. rust-red.
36	1935·3	2·5	2·2	17	SEB. pinkish, NEB. brownish.
37	1936·6	0·0	5·4	33	NEB. conspicuous rust-red.

The last observation calls for some additional remarks. The NEB. was very broad and very dark and nearly uniform in density, whilst its rust-red colour was most conspicuous, especially in August and September, when its redness was often rated at 6 or even 7 of the scale of redness. The belt was particularly red about the longitude of the red spot. The SEB., on the other hand, was very faint, on many nights indeed too faint for colour in my little telescope. When best seen it appeared devoid of all red colour, but rather greyish or even bluish grey in tint. The conspicuous red colour of the NEB. has been remarked by other observers. For instance by Quénisset at Juvisy and Antoniadi at Meudon. The latter describes the coloration as being "rose violacée."* He does not mention any colour in connection with the SEB.

The additional observations are shown in diagram form on the right-hand side of the adjoining figure, which also includes on the left the last part of

* *l'Astronomie*, **50**, 380, 1936.

fig. 1 of the second paper, in order to show the connection between the two diagrams. It will be seen that we have now another of the familiar lozenge-shaped interspaces, yielding the date for another of what have been termed



Colour changes of Jupiter's Belts.

Standard Epochs, namely 1932·2. This date is considerably earlier than the one anticipated from the previously adopted period of 12·5 years.

Table II gives the observed and computed dates of the 7 Standard Epochs now available, the latter from the elements of variation mentioned below. No. 8 is the new one, but the others are the same as before with the exception

TABLE II

Observed and Computed Standard Epochs

E	Observed Epoch	Computed Epoch	*O* − *C*
0	1837·0	1838·52	−1·52
3	1873·3	74·10	−0·80
4	1885·5	85·96	−0·46
5	1897·7	97·82	−0·12
6	1912·2	1909·68	+2·52
7	1923·1	21·54	+1·56
8	1932·2	33·40	−1·20

of the first, which has been changed from 1836·0 to 1837·0. The reason for this is that the lozenge-shaped interspace was in this case only half observed, as will be seen from the diagram on p. 380 of the first paper, whilst the mean colour curve (p. 704 of the second paper) indicates a later date. The change is not of much importance at the present time, there being more and better observations now available.

If we compare the two extreme observations, 1837·0 and 1932·2, the value of P (the period) comes out to be 11·9 years. Similarly, from the extreme dates of the maximum redness of the NEB. P is 11·64 years, and from the extreme dates of the minimum redness of the SEB. P is 11·68 years. The mean of these three comparisons is 11·74 years, so near to the period of the planet's revolution round the Sun (11·86 years) that there can now be no hesitation in assuming the latter as the mean period of the changes in redness of the two belts. The following elements of variation have therefore been adopted :—

Years
Standard epoch = 1838·52 + 11·86 E.

The residuals in the last column of Table II still, perhaps, show a slight tendency towards a longer period than 11·86 years, but it is so slight that it may be disregarded. It is well within the apparent errors of observation.* It is to the abnormal length of the lozenge-shaped interval E. 6 that the longer period of 12·5 years, derived in the second paper, must be ascribed. This has now been almost counterbalanced by the shorter lengths of the intervals E. 7 and E. 8. The residuals seem to suggest the possible existence of a secondary periodicity superimposed upon the principal one of 11·86 years.

Notwithstanding the small inclination of the plane of Jupiter's equator to the plane of his orbit and his great distance from the Sun, it is evident that the periodic changes in the colours of the two belts are to be ascribed to seasonal changes on the planet itself. These changes are unrelated to the well-known 11-year sunspot periodicity.

Some Final Conclusions

1. Additional observations have led to a better determination of the elements of variation.

2. It is now certain that the period of variation is in the long run the same as that of the planet's revolution round the Sun (11·86 years).

3. The adopted elements of variation are :—

Years
Standard epoch = 1838·52 + 11·86 E.

4. At the times of a Standard Epoch the NEB. is at or near a minimum of redness, whilst the SEB. is at or near a maximum.

5. At the intermediate epochs the colours of the belts are reversed from those above mentioned.

* Which include the minor real or temporary fluctuations in the redness of the belts.

6. The computed date of the next Standard Epoch is 1945·2 when the NEB. will be at or near a minimum of redness and the SEB. at or near a maximum.

Observations of the redness of the two equatorial belts in 1945, and during a year or two before and after that date, are much to be desired, and it is to be hoped that such will be forthcoming from other observers. The work is quite simple and interesting. All that is necessary is to estimate the redness of the two belts, if not on the adopted scale, in simple words, such as colourless (or bluish), slightly red, red, deep red, intense red, and so on. Fancy descriptions of the colours had best be avoided. The early descriptions of the colours of double stars demonstrate the inutility of this.

Note on the Red Spot

The great plainness of the red spot during the past opposition will have been noticed by many. It was first brought to my knowledge in a letter from Mr. Peek (Director of the Jupiter Section of the B.A.A.), and it was soon confirmed by direct observation with the telescope. According to my recollection the spot has been, in fact, decidedly plainer than at any time previously since about the year 1883. The red colour, too, has been very conspicuous, attaining to 5 or even 6 on the scale of redness (which ranges from 0 to 10). Antoniadi at Meudon describes the colour as a very intense "rose vermillion." It usually seemed to me less red than the NEB. Quénisset's interesting photographs (*l'Astronomie*, **50**, 45, 1936) give a very good idea of the appearance of the planet and the spot as viewed with a very low power on a very bad night. Under such conditions the appearance was almost exactly the same as depicted in an old drawing by Cassini, dated about the year 1665.

Polvarth, St. Mawes :
1936 November 2.

ON THE POSSIBLE RESULTS OF AN ENCOUNTER OF PLUTO WITH THE NEPTUNIAN SYSTEM.

Raymond A. Lyttleton.

Introduction.—The changes which occur in a planetary orbit due to the action of the other planets indicate that, except in the case of Pluto, the general features and extent of the solar system remain more or less unaltered over a very long interval. But Pluto may at times be nearer to the Sun than Neptune, so that as the nodes and apses of their orbits move the corresponding elliptic paths may ultimately intersect. Thus encounters of Pluto are clearly relevant to the development, both past and future, of the solar system.

1. During an encounter of the two planets the action of the Sun will be neglected and the presence of Triton ignored. The validity of this approxi-

mation to the motion is manifest, but it may also be used as a definition of a close encounter.

On this basis the relative velocity of the two bodies is unaltered in magnitude by an encounter, and hence the effects will be greatest (*ceteris paribus*) when the relative velocity is largest. Since the orbit of Neptune is so nearly circular this maximum value depends almost entirely on the angle of intersection of the orbits and occurs when this angle is greatest.

For Neptune: Mass $= \odot \div 19{,}314$
$a = 30{\cdot}07$ units, $e = 0{\cdot}0085$, $i = 1^\circ\ 47'$.

For Pluto: Mass $= \odot \div 3 \times 10^6$ approximately,
$a = 39{\cdot}60$ units, $e = 0{\cdot}2537$, $i = 17^\circ\ 9'$.

Taking Neptune's orbit to be circular, i the angle between the orbital planes and ψ the angle which Pluto's path makes with a coplanar circle through the point in question, the cosine of the angle between the paths is $\cos i \cos \psi$. The maximum values of i and ψ are $18^\circ\ 56'$ and 5° approximately, so that for the present values of the eccentricity and inclination the maximum angle is about $19^\circ\ 24'$. (The error due to the neglect of Neptune's orbital eccentricity is about $\frac{1}{2}^\circ$.)

Consider first an encounter near the aphelion of Neptune. The velocity of Neptune at aphelion is

$$v \text{ (Neptune)} = 0{\cdot}1808 \text{ astron. unit of velocity} = 3{\cdot}344 \text{ miles per sec.}$$

At this distance

$$v \text{ (Pluto)} = 0{\cdot}2018 \text{ unit} = 3{\cdot}731 \text{ miles per sec.},$$

while the velocity of escape at this distance is

$$v \text{ (escape)} = 0{\cdot}2568 \text{ unit} = 4{\cdot}749 \text{ miles per sec.}$$

And hence the maximum relative velocity is

$$v \text{ (relative)} = 0{\cdot}0676 \text{ unit} = 1{\cdot}253 \text{ miles per sec.}$$

Now if d is the distance of closest approach of two bodies in hyperbolic orbits and W their relative velocity before encounter, the eccentricity of the orbits is $1 + W^2 d/(\text{Mass})$. Taking d to be 20,000 miles, the value of the eccentricity is less than 1·05, so that the orbit can be almost parabolic and the relative velocity may be turned through nearly 150°.

The greatest velocity referred to the Sun which Pluto may have after such an encounter corresponds to the relative velocity being in the same direction as that of Neptune in its orbit, and neglecting the mass of Pluto this value is

$$3{\cdot}344 + 1{\cdot}253 = 4{\cdot}597 \text{ miles per sec.},$$

and this is less than the escape velocity 4·749 miles per sec. The corresponding angle through which the relative velocity must be turned is easily found to be

$$\sin^{-1}\left(\frac{3{\cdot}731}{1{\cdot}253} \sin 19^\circ\ 24'\right) \text{ or } 81^\circ\ 18'.$$

Hence the eccentricity of the encounter orbit is cosec 41° 9′ or 1·52. Accordingly the requisite distance to which Pluto must approach is $\frac{\text{Mass} \times 0{\cdot}52}{W^2}$, which is

0·006 astron. unit = 550,000 miles

or about two and a half times the present distance of Triton from the planet.

The greatest effect on the major axis of the orbit of Pluto therefore takes place for an encounter at quite a considerable distance from Neptune. It is also of interest to notice that, as far as Pluto is concerned, the mass of Neptune is not involved so long as it is large compared with Pluto. However, the distance at which the greatest effect is produced increases with the mass so that a larger planet is more likely to produce this greatest effect. The new semi-major axis is found to be 240 astronomical units, corresponding to a period of over 3700 years.

Similarly if Pluto encounter Neptune at the perihelion of the latter,

v (Neptune) = 3·404 miles per sec.,
v (Pluto) = 3·781 miles per sec.,
v (escape) = 4·788 miles per sec.,

and the maximum relative velocity is 1·269 miles per sec. Hence the velocity of Pluto after encounter cannot exceed 4·673 miles per sec. The semi-major axis of the orbit may therefore be as great as 314 units and the period as long as 5560 years.

The maximum and minimum distances from the Sun can also be shown to occur only for the velocities in the direction of the orbital motion of Neptune—that is, for the maximum and minimum velocities (see § 2). The nearest approach to the Sun corresponds to the smallest velocity and is found to be 7·3 astronomical units, while the greatest distance is 598 astronomical units.

Thus the orbit of Pluto may come within that of Saturn, and it can be shown that it is possible for an encounter with Saturn to effect the ejection of Pluto from the system. But such an encounter is of little interest on account of its extreme unlikelihood.

These remarks indicate the possibilities of Pluto's orbit so long as it is affected only in the manner here suggested. In so far as these direct changes due to Neptune are concerned it is particularly to be noticed that, however many such changes occur in the orbit of Pluto, the relative velocity of the two bodies (before or after an encounter) is unaltered in magnitude to the order of approximation used here. Hence at no time may Pluto be ejected from the system and in all probability is therefore an original member. But the orbit of Pluto may slowly undergo changes of eccentricity and inclination with a possible increase of the relative velocity to an extent permitting Pluto to be expelled.

2. To find the maximum and minimum distances from the Sun of a particle projected with relative velocity U from a point moving with velocity V perpendicular to the direction of the Sun and at a distance R from it.

Supposing U to make an angle θ with V, the energy and angular momentum give at once

$$\frac{1}{a}=\frac{2}{R}-(V^2+2VU\cos\theta+U^2)=x$$

$$a(1-e^2)=R(V+U\cos\theta)$$

for the subsequent orbit. Hence

$$a(1-e)=\frac{1-\sqrt{1-\lambda x(\mu-x)^2}}{x}=L,$$

where

$$\lambda=\frac{R^2}{4V^2} \quad \text{and} \quad \mu=\frac{2}{R}+V^2-U^2.$$

For this distance to be least $\frac{dL}{d\theta}=0$, and this occurs if

$$\frac{dx}{d\theta}=0 \quad \text{or} \quad \sin\theta=0,$$

giving the critical values corresponding to the greatest and least velocities, or $\frac{dL}{dx}=0$ giving

$$x^2(\mu-x)[(\mu-x)(\mu+x)^2-8/\lambda]=0.$$

In the present connection, for Pluto $V>U$ and hence $0<x<\mu$. Putting

$$x+\mu=y \quad \text{then} \quad \mu<y<2\mu \quad \text{and} \quad y^2(y-2\mu)=-8/\lambda.$$

The numerical value of μ is 0·091 and of $8/\lambda$ is 0·036, so that

$$y^2(y-0\cdot192)=-0\cdot036,$$

and this equation clearly has no root between 0·091 and 0·192. Accordingly critical values of the distance of the apses occur only for the maximum and minimum velocities.

More generally the left-hand side has stationary values when $3y^2-4\mu y=0$. There will therefore be two positive roots if $4\lambda\mu^3>27$, and these will lie on opposite sides of $4\mu/3$, and one at least of these necessarily gives a real value of θ. The negative root does not correspond to a real θ.

The hyperbolic case may be discussed by writing $-x$ for x.

3. So far the mass of Pluto has been ignored. The similarity in brightness of Pluto and Triton represents almost the only certain physical knowledge of these bodies, and according to Nicholson the mass of the latter is about one-tenth of that of the Earth. Adopting a similar value for Pluto,

$$\text{Pluto}=\text{Earth}\times0\cdot1=\text{Neptune}\times0\cdot006.$$

After an encounter at the perihelion of Neptune the extreme velocities which Neptune may have are approximately

$$3{\cdot}404 \pm (1{\cdot}269 \times 0{\cdot}006) = 3{\cdot}412 \text{ and } 3{\cdot}396 \text{ miles per sec.},$$

corresponding to semi-major axes,

30·26 and 29·98 units,

while the periods are

166·5 and 164·2 years.

Thus the period could be changed by over a year.

4. Encounters between Pluto and Triton.

The relative velocity of Pluto and Neptune before an encounter has been seen to be at most 1·26 miles per sec. The orbital velocity of Triton is 2·72 miles per sec. and the escape velocity at its distance is 3·84 miles per sec. For a particle in an elliptic orbit near Neptune

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right),$$

so that for Triton $\mu/a = (2{\cdot}72)^2$. For a particle in a hyperbolic orbit with the velocity 1·26 at infinity

$$v^2 = 2\mu/r + (1{\cdot}26)^2,$$

so that at the distance of Triton (due to the action of Neptune only) the maximum velocity of Pluto would be

$$\sqrt{2 \times (2{\cdot}72)^2 + (1{\cdot}26)^2} = 4{\cdot}05 \text{ miles per sec.},$$

while the least velocity would exceed 3·84 miles per sec.

The investigation of the encounters will be made by neglecting the action of Neptune and of course of the Sun. On account of the small mass of Pluto and Triton this is not a very good approximation.

The eccentricity of the relative orbit is $1 + \dfrac{W^2 d}{\text{Mass}}$, where d is the closest distance of approach of the two bodies and W the relative velocity before or after. Taking d to be 5000 miles and the mass = 0·2 Earth, then $\dfrac{d}{\text{Mass}} = 90$ (using astron. units). W has limits $4{\cdot}05 + 2{\cdot}72$ and $3{\cdot}84 - 2{\cdot}72$ or 6·77 and 1·12 miles per sec. Hence in astronomical units W^2 ranges from 0·134 to 0·0037, thus e ranges from 13 to 1·33. If the masses be equal the changes of velocity (vectorially) which may be communicated may vary from $W/13$ for large relative velocities to $W/1{\cdot}33$ for small relative velocities. While the angle through which the direction of the velocity relative to Neptune can be turned lies between 9° and 97°, so that the asymptotes are inclined at not less than 83°.

The following diagrams indicate some of the more interesting encounters which could occur. The circles represent the spheres of equal velocity relative to Neptune. Thus the outermost sphere has radius equal to the velocity of Pluto, just exceeding the escape velocity, which is the radius of the next sphere, while the smallest sphere has radius equal to the velocity of

Triton in its orbit. The diagrams are to be interpreted as three-dimensional, but it is to be noticed that the extreme cases occur in the plane of Triton's motion. T and P are the ends of the vectors representing the velocities relative to Neptune of Triton and Pluto before encounter in a typical case. The velocity of the centre of mass G of Pluto and Triton is unaltered by the

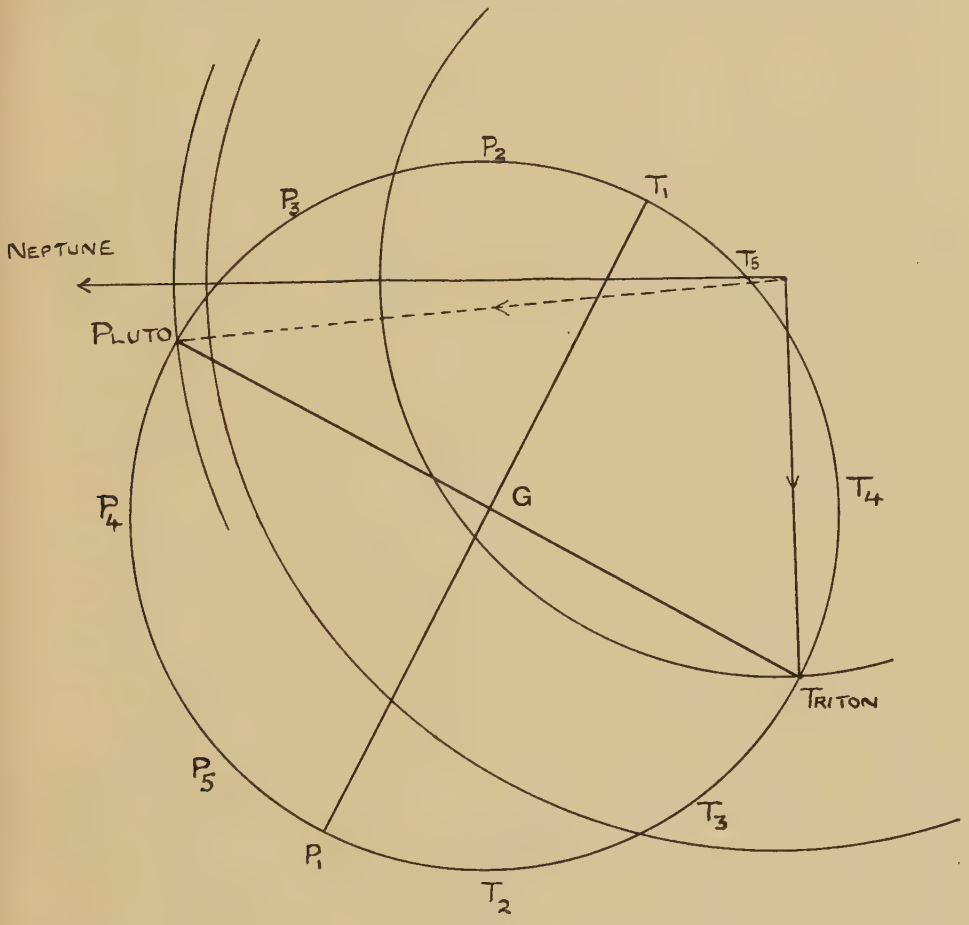


FIG. 1.

encounter, while the relative velocity of the bodies is changed only in direction. Hence the subsequent velocities may be derived from the initial motion simply by a rotation of TP about any line through G.

The following results may occur (they are shown in fig. 1 by the corresponding suffix) :—

(i) Triton sent into a smaller direct orbit of high eccentricity T_1 and Pluto not captured P_1.

(ii) Triton ejected and Pluto captured in a direct orbit.

(iii) Triton's orbit disturbed but not radically altered T_3 and Pluto captured in a direct orbit P_3.

(iv) Triton's orbit disturbed T_4 and Pluto not captured.

(v) The angular momentum of Triton reduced to zero T_5 so that it falls into Neptune, Pluto not captured P_5.

(vi) It is also possible that if Pluto had initial velocity in the same general direction as that of Triton, Pluto could be captured in a retrograde orbit with little change in that of Triton. (Not shown in figure.)

5. There is a further very remarkable case. If a closer encounter is

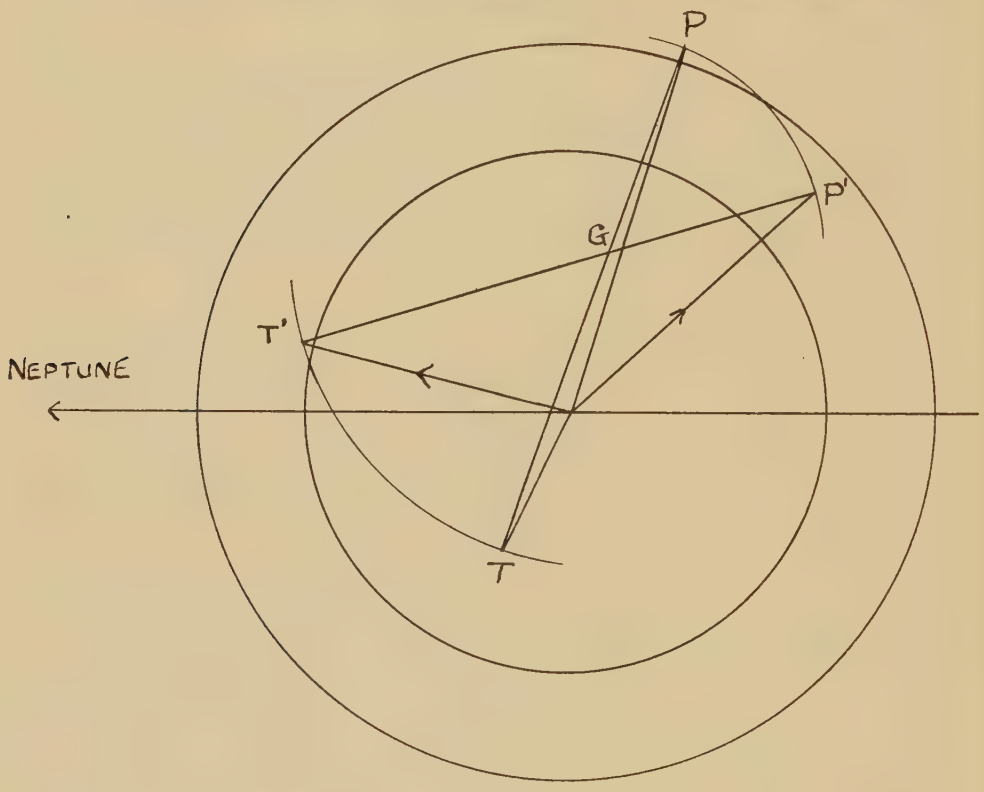


FIG. 2.

permissible the value of the eccentricity of the encounter orbit may be reduced even though the relative velocity is fairly large. The eccentricity is also considerably reduced if the encounter occurs at a distance from Neptune greater than the present distance of Triton. In these circumstances if the velocity of Pluto is in practically the opposite direction to that of Triton the second figure shows that an encounter may result in the capture of Pluto in a direct orbit and the reversal of the general direction of Triton's motion, so that Neptune would have two direct satellites. Now such a motion is clearly capable of complete reversal, so that the interesting possibility arises that Pluto may have been a satellite of Neptune. The encounter which would effect this is rather extreme, but only if it is required that Triton be thrown into a circular retrograde orbit from the start ; a relaxation of this condition facilitates the reversal of the orbital motion. The

figure is drawn for such a case and also with a view to demonstrating that if Pluto's mass exceeds that of Triton the result is more readily accomplished. The eccentricities of the direct satellite orbits are seen to be large, but if two satellites were describing orbits which permitted a very close encounter there seems little doubt that the perturbations would be very great long before an encounter as close as is envisaged here actually occurred, and these might give rise to eccentric orbits.

The suggestion is supported by the following considerations. Supposing Neptune to have had two direct satellites, a mechanism has to be provided to bring them to a stage where encounter might occur. Now tidal friction, if it were very great, might cause the mean distances of the satellites to approach equality with a resulting encounter. If Pluto had always been an independent planet it would be natural to expect that its orbit would lie completely outside that of the others. (Jeffreys's theory of the rounding-up of orbits by a resisting medium requires a much longer time for a given effect to be produced for Pluto than for Neptune; but the theory is uncertain for massive planets.) Thus Pluto may have been comparatively recently ejected from the Neptunian system. This is again accordant; for tidal friction may have taken place late in the history of the planet, and, furthermore, it is noticeable in this connection that the rotation period of Neptune is by far the longest of the major planets.

Summary.—(i) Encounters of Pluto with the Neptunian system may eventually occur and may have occurred in the past. Their investigation throws some light on the anomalies at the edge of the solar system.

(ii) Under Neptune's action Pluto cannot be ejected from the system, but the extent of its orbit may be greatly altered.

(iii) If Pluto should encounter Triton several interesting cases arise, but these are of slight interest owing to their improbability.

(iv) Nevertheless it appears from these considerations that Pluto may originally have been a direct satellite of Neptune, and that the encounter which gave it an existence as an independent planet also reversed the general direction of motion of Triton, thus giving a second, though rather speculative, possible explanation of the retrograde motion of a true satellite.*

It is a pleasure to accord my thanks to Professor H. N. Russell for the benefit of many discussions of the present work.

Princeton University Observatory:
1936 *October* 30.

* *M.N.*, 96, 566, 1936.

THE STELLAR PROBLEM OF THREE BODIES: III. THE MOTIONS OF THE APSE AND NODE WITH APPLICATIONS TO THE MOON.

E. W. Brown, Sc.D., F.R.S.

1. The chief result in this paper is an unexpected by-product, namely, the reconciliation of the theoretical and observed values of the motion of the lunar perigee. There had been an outstanding difference of 17″ in the centennial motion, and this was three or four times the estimated probable error. Two-thirds of it is accounted for by the terms with factor e'^4, which I had not computed, because the first term arranged along powers of m, given by Delaunay, contributed only 1″. Below I find that the second term has a factor of 40 and contributes 3″, and it is not difficult to estimate that the whole series will add from 9″ to 13″. The terms of the sixth order with respect to e, γ, which were computed neither by Delaunay nor myself, add some 3″ more.

The motion of the node is practically unaffected by the terms in e'^4, but the sixth order terms add about 4″, which increases the discrepancy from $-12''$ to $-16''$. This amount, however, is not much greater than the probable error of the comparison.

Since the observed values were used in the lunar tables the latter are not affected.

2. In the second paper I found the chief periodic terms which, if any, would be observable. In this paper the motions of the apse and node are obtained as far as m^3, where m is the ratio of the periods. Although these will possibly not be observable in any system for many years to come, it was of some interest to compute the magnitudes. In trying to do so without the use of infinite series, which generally limit the eccentricities and inclination to small values, I came across three or four mathematical problems each of which required some special device for its solution. The most interesting of these is the one which shows that the terms depending on the eccentricity of the orbit of the outer body can be deduced from those independent of this parameter by the application of a finite factor different for each power of m. Only the terms depending on the second and third powers of m are computed, but these contain in each case all powers of the eccentricities and inclination.

The details of the work appeared in a form which is not usual in the problems of celestial mechanics, and I was glad to be able to check a considerable portion of the results by comparison with the literal developments given by Delaunay for the motions of the perigee and node of the Moon.

3. Another side to the work is the possibility that it may open up a method of treatment of satellite motion when the eccentricities and inclination are large. One noticeable feature is the important part played by the

terms in the motion whose arguments depend only on the apse and node and especially by the term which has twice the difference of the angle between the apse and node as its argument. The chief limitation is the requirement that the ratio of the periods shall be small, but this is the case with many of the satellites in the solar system, so that in these cases it will not be a serious bar to more knowledge of their possible motions. I have, however, left these problems untouched in the present paper.

4. *The Canonical Equations of Motion.*—In Paper I it was shown that if we neglect the short-period terms in the portions of the disturbing function which contain the long-period and apse-node terms, the errors of the coefficients of the latter in this function will have the factor m^4 where m is the ratio of the periods. This fact enables us to find the coefficients of the long-period terms in the elements to the order m^2, and of the apse-node terms to the order m, and the mean motions of the apse and node to the order m^3, by simply neglecting all short-period terms. As no expansions into infinite series of the eccentricities and inclination are made, the disturbing function is complete under the adopted limitations. The second term of R will be neglected; since it contains a/a' as a factor, its effect on the mean motions will have the factor m^2a^2/a'^2, and this is outside the range of our work. With the independent variable f', which has replaced t, the powers of m mentioned above are reduced by one power.

I use the Delaunay variables L, G, H, l, g, h defined in Paper II. Since l is absent we have L constant, and the equations are reduced to

$$\frac{dG}{df'}=\frac{\partial R'}{\partial g},\qquad \frac{dg}{df'}=-\frac{\partial R'}{\partial G},\qquad \frac{dH}{df'}=\frac{\partial R'}{\partial h},\qquad \frac{dh}{df'}=-\frac{\partial R'}{\partial H},$$

where

$$R'=k_1(1+e'\cos f')\left[\left\{\left(\frac{5}{2}-\frac{3}{2}\frac{G^2}{L^2}\right)\left(-\frac{1}{3}+\frac{H^2}{G^2}\right)+\frac{5}{2}\left(1-\frac{G^2}{L^2}\right)\left(1-\frac{H^2}{G^2}\right)\cos 2g\right\}\right.$$
$$\left.+\left(\frac{5}{2}-\frac{3}{2}\frac{G^2}{L^2}\right)\left(1-\frac{H^2}{G^2}\right)\cos(2f'+2\varpi'-2h)\right.$$
$$\left.+\tfrac{5}{4}(1-G^2)\left\{\left(1+\frac{H}{G}\right)^2\cos(2f'+2\varpi'-2g-2h)+\left(1-\frac{H}{G}\right)^2\cos(2f'+2\varpi'+2g-2h)\right\}\right],$$

with

$$k_1=\frac{3}{8}\frac{n'}{n}\cdot L(1-e'^2)^{-\frac{3}{2}}.$$

The long-period terms are those containing the argument f' and the apse-node terms those containing g or h, but independent of f'. It is supposed that R' is expressed as a sum of cosines of multiples of f', g, h, with coefficients depending on L, G, H.

5. *Elimination of the Long-period Terms.*—If ϕ be any function of f' and the elements, we have

$$\frac{d\phi}{df'}=\frac{\partial\phi}{\partial f'}+\frac{\partial\phi}{\partial G}\cdot\frac{dG}{df'}+\ \ldots\ +\ \ldots\ +\frac{\partial\phi}{\partial h}\cdot\frac{dh}{df'},$$

or, written in the form of operators,

$$D = D_f + D_e,$$

where D_f denotes the partial derivative with respect to f', and D_e the portion which arises through the variation of the elements. Since the canonical equations show that dG/df', . . . contain the factor k_1 it follows that the effect of the operator D_e is to introduce a factor k_1. The expansion

$$\frac{1}{D} = \frac{1}{D_f + D_e} = \frac{1}{D_f} - \frac{1}{D_f^2} D_e + \frac{1}{D_f^3} D_e^2 - \ldots$$

is therefore equivalent to an expansion in powers of k_1, care being taken concerning the order of the operators.

Consider any one of the canonical equations. Divide R' into two portions, one, R_c, that independent of f', and the other, R_f, that which contains the terms having f' in their arguments. We have

$$\begin{aligned} G &= \frac{1}{D}\frac{\partial R_c}{\partial g} + \frac{1}{D}\frac{\partial R_f}{\partial g} = \frac{1}{D}\frac{\partial R_c}{\partial g} + \frac{1}{D_f}\frac{\partial R_f}{\partial g} + \left(\frac{1}{D} - \frac{1}{D_f}\right)\frac{\partial R_f}{\partial g} \\ &= \frac{1}{D}\frac{\partial R_c}{\partial g} + \frac{1}{D_f}\frac{\partial R_f}{\partial g} - \frac{1}{D}\left(D_e \frac{1}{D_f}\frac{\partial R_f}{\partial g}\right). \end{aligned}$$

That the order of the operators in the last term is correct is seen by operating on the equation with D.

This last term is divided into two portions—that independent of f' and denoted by the suffix c and that depending on f' and denoted by the suffix f. It is evident that we now have a formula which can be continued indefinitely in the same manner. We obtain

$$\begin{aligned} G = &\frac{1}{D}\left[\frac{\partial R}{\partial G} - D_e \frac{1}{D_f}\frac{\partial R_f}{\partial g} + D_e \frac{1}{D_f}\left(D_e \frac{1}{D_f}\frac{\partial R_f}{\partial g}\right)_f - \ldots\right]_c \\ &+ \frac{1}{D_f}\left[\frac{\partial R}{\partial G} - D_e \frac{1}{D_f}\frac{\partial R_f}{\partial g} + D_e \frac{1}{D_f}\left(D_e \frac{1}{D_f}\frac{\partial R_f}{\partial g}\right)_f - \ldots\right]_f. \end{aligned}$$

The expression in the second line consists wholly of integrations in which f' is the only variable, so that the terms independent of f' arise solely from the first line. The first term of the second line was given in Paper II, and the other terms will not be needed, so that attention can now be confined to the terms in the first line.

6. *Reduction of the Second of the Terms Independent of f'.*—For brevity put

$$\frac{1}{D_f} R_f = S.$$

Then since the partial derivatives with respect to f' and the elements are commutative, we have

$$\frac{1}{D_f}\frac{\partial R_f}{\partial G} = \frac{\partial S}{\partial G}.$$

Now

$$D_e\frac{\partial S}{\partial G}=\frac{\partial^2 S}{\partial G^2}\frac{dG}{df'}+\frac{\partial^2 S}{\partial G\partial g}\frac{dg}{df'}+\frac{\partial^2 S}{\partial G\partial H}\frac{dH}{df'}+\frac{\partial^2 S}{\partial G\partial h}\frac{dh}{df'}$$
$$=\frac{\partial^2 S}{\partial G^2}\frac{\partial R}{\partial g}-\frac{\partial^2 S}{\partial G\partial g}\frac{\partial R}{\partial G}+\frac{\partial^2 S}{\partial G\partial H}\frac{\partial R}{\partial h}-\frac{\partial^2 S}{\partial G\partial h}\frac{\partial R}{\partial H}.$$

Adopt the notation

$$(p, q)=\frac{\partial p}{\partial G}\frac{\partial q}{\partial g}-\frac{\partial p}{\partial g}\frac{\partial q}{\partial G}+\frac{\partial p}{\partial H}\frac{\partial q}{\partial h}-\frac{\partial p}{\partial h}\frac{\partial q}{\partial H}=-(q, p).$$

The previous equation can then be written

$$D_e\frac{\partial S}{\partial G}=\left(\frac{\partial S}{\partial G}, R\right).$$

But

$$\frac{\partial}{\partial f'}\left(\frac{\partial S}{\partial G}, S\right)=\left(\frac{\partial R_f}{\partial G}, S\right)+\left(\frac{\partial S}{\partial G}, R\right),$$
$$\frac{\partial}{\partial G}(R, S)=\left(\frac{\partial R_f}{\partial G}, S\right)+\left(R, \frac{\partial S}{\partial G}\right),$$

whence

$$\left(\frac{\partial S}{\partial G}, R\right)=\frac{1}{2}\frac{\partial}{\partial f'}\left(\frac{\partial S}{\partial G}, S\right)+\frac{1}{2}\frac{\partial}{\partial G}(S, R).$$

This equation may also be written in the form

$$-\delta\frac{\partial R_f}{\partial G}=\left(\frac{\partial S}{\partial G}, R\right)=\frac{1}{2}\frac{\partial}{\partial f'}\left(\delta\frac{\partial S}{\partial G}\right)-\frac{1}{2}\frac{\partial}{\partial G}(\delta R),$$

where δ denotes the portions additional to $\partial S/\partial G$, R due to inserting in these functions the portions of the variable parts of the elements. This theorem shows the importance of remembering that $\delta(\partial R/\partial G)$ is not equal to $\partial(\delta R)/\partial G$.

The first of these terms contains no portion independent of f' and is therefore excluded. Thus the portions independent of f' to the order k_1^2 are obtained by solving the canonical equations with

$$R'=R_c+\tfrac{1}{2}(\delta R_f)_c=R_c+\tfrac{1}{2}(R_f, S)_c,$$

since $(\partial S/\partial G, R_c)$ contains no term independent of f'.

7. *Elimination of* e'.—The disturbing function contains only three multiples of f', and the portions independent of f' in (R_f, S) are obtained by taking the terms in R_f having a given multiple of f' with the corresponding terms in S.

The terms containing the first multiple of f' in R, S can be written in the form

$$R=A\cos f'+B\cos(f'+2\varpi'-2h)+C\sin(f'+2\varpi'-2h),$$
$$S=A\sin f'+B\sin(f'+2\varpi'-2h)-C\cos(f'+2\varpi'-2h),$$

where A, B, C are functions of G, g, H and contain the factor e'.

The combination of the first terms of R, S in (R, S) give $(A, A)\cos f' \sin f'$, which is zero. The combination of the first with the second or third terms gives terms containing the argument $2\varpi' - 2h$, and it will be seen later on that these terms can produce no constant part with the factor k_1^2. It follows that for the calculation of $(R_f, S)_c$ we need only those terms containing the argument $2\varpi' - 2h$.

When $e' = 0$, these portions of R can be written $R = \psi(2f' + 2\omega' - 2h)$, and when $e' \neq 0$, the required parts of R are

$$R = \psi(2f' + 2\varpi' - 2h) + \tfrac{1}{2}e'\psi(f' + 2\varpi' - 2h) + \tfrac{1}{2}e'\psi(3f' + 2\varpi' - 2h),$$

so that the corresponding parts of S are

$$S = \tfrac{1}{2}\psi_1(2f' + 2\varpi' - 2h) + \tfrac{1}{2}e'\psi_1(f' + 2\varpi' - 2h) + \tfrac{1}{6}e'\psi_1(3f' + 2\varpi' - 2h),$$

where $\psi = d\psi_1/df'$. It follows that in finding the portion of (R_f, S) independent of f', the ratio of its value when $e' \neq 0$ to that when $e' = 0$ is

$$\tfrac{1}{2} + \frac{e'^2}{4} + \frac{e'^2}{12} : \tfrac{1}{2} = 1 + \tfrac{2}{3}e'^2 : 1.$$

If, therefore, this factor be inserted, we can put $e' = 0$ in R.

Since k_1 contains the factor $(1 - e'^2)^{\frac{3}{2}}$ and since these terms have the factor k_1^2, we have the following new theorems :—

The terms containing e' in the mean motions of the apse and node and factored by m^2 are derived from those independent of e' by multiplying the latter by the factor $(1 - e'^2)^{-\frac{3}{2}} - 1$, and those which are factored by m^3 from those independent of e' by multiplying the latter by $(1 + \frac{2}{3}e'^2)(1 - e'^2)^{-3} - 1$. These theorems include all powers of e, γ, where γ is the sine of half the inclination.

Written as series, these factors are

$$\tfrac{3}{2}e'^2 + \tfrac{15}{8}e'^4 + \tfrac{35}{16}e'^6 + \ldots, \qquad \tfrac{11}{3}e'^2 + 8e'^4 + 14e'^6 + \ldots$$

It is possible that similar finite factors exist for the higher powers of m, but I see no way of obtaining them without a very considerable amount of algebraic work. It is, however, evident that the effect of e' is an increase in the forward motion of the apse. The node is comparatively little affected by the third and higher powers of m, when the ratio of the periods is small, and their signs may be either positive or negative.

8. *Elimination of Derivatives with respect to h.*—A final abbreviation of the work of computing $(R_f, S)_c$ arises from the fact that with the only terms retained

$$\frac{\partial R}{\partial h} = -\frac{\partial R}{\partial f'} = 4S, \qquad \frac{\partial S}{\partial h} = -\frac{\partial S}{\partial f'} = -R,$$

so that

$$\frac{\partial R}{\partial H}\frac{\partial S}{\partial h} - \frac{\partial R}{\partial h}\frac{\partial S}{\partial H} = -R\frac{\partial R}{\partial H} - 4S\frac{\partial S}{\partial H} = -\frac{1}{2}\frac{\partial}{\partial H}(R^2 + 4S^2).$$

The expressions for R_f, S are reduced to

$$R = \alpha\cos\phi + \beta\sin\phi, \qquad S = \tfrac{1}{2}\alpha\sin\phi - \tfrac{1}{2}\beta\cos\phi,$$

where

$$\alpha = k_1\left(\frac{5}{2} - \frac{3}{2}\frac{G^2}{L^2}\right)\left(-\frac{1}{3} + \frac{H^2}{G^2}\right) + \frac{5}{2}k_1\left(1 - \frac{G^2}{L^2}\right)\left(1 + \frac{H^2}{G^2}\right)\cos 2g,$$

$$\beta = k_1\left(\frac{5}{2} - \frac{3}{2}\frac{G^2}{L^2}\right)\left(-\frac{1}{3} + \frac{H^2}{G^2}\right) - \frac{5}{2}k_1\left(1 - \frac{G^2}{L_2}\right)2\frac{H}{G}\sin 2g,$$

with $\phi = 2f' + 2\varpi' - 2h$, and therefore

$$(R_f,\ S)_c = \frac{\partial\alpha}{\partial G}\frac{\partial\beta}{\partial g} - \frac{\partial\alpha}{\partial g}\frac{\partial\beta}{\partial G} - \frac{1}{2}\frac{\partial}{\partial H}(\alpha^2 + \beta^2).$$

The computation is now quite simple. The term with argument $4g$ disappears, and I find

$$\tfrac{1}{2}(R_f,\ S)_c = \tfrac{1}{2}k_1{}^2\frac{H}{L^2}\left\{35 - 33\frac{G^2}{L^2} + 15\frac{H^2}{G^2} - 17\frac{H^2}{L^2} + 15\left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\cos 2g\right\}.$$

This is to be added to R_c, which is

$$R_c = \tfrac{1}{2}k_1\left\{\frac{G^2}{L^2} + 5\frac{H^2}{G^2} - 3\frac{H^2}{L^2} + 5\left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\cos 2g\right\}.$$

Since the constant term in R_c has no effect on the motion we can write

$$\frac{k_1}{L}R = R_c + \tfrac{1}{2}(R_f,\ S)_c = \left(1 + 3\frac{Hk_1}{L}\right)R_c + \tfrac{1}{2}k_1{}^2\frac{H}{L^2}\left(35 - 36\frac{G^2}{L^2} + 8\frac{H^2}{L^2}\right).$$

These expressions show what may be expected in the relative ratios of convergence of g, h along powers of m. The principal parts of the additional terms are approximately in the ratio of 24 : 1, so that the addition to the motion of the apse is 25 times that to the motion of the node.

9. *Solution of the Equations.*—Since l is not present in R we have $L = \text{const.}$ Further, since k_1 is a factor of R, we can change the independent variable by putting

$$\tau = \frac{k_1}{L}f',$$

and the first term of the new $R = \text{old } R \div k_1$ is independent of k_1.

Next, since h is not present in R we have $H = \text{const.}$ and the canonical equations reduce to

$$\frac{dG}{d\tau} = \frac{\partial R}{\partial g}, \qquad \frac{dg}{d\tau} = -\frac{\partial R}{\partial G}, \qquad \frac{dh}{d\tau} = -\frac{\partial R}{\partial H},$$

the last equation being integrable as soon as we have found the values of G, g from the first two equations.

Consider the first term only of R so that the equations are independent of k_1. Their solution will have the form

$$G = \Sigma G_i \cos 2i(g_c\tau + g_0), \qquad g = g_c\tau + g_0 + \Sigma g_i \sin 2i(g_c\tau + g_0),$$

where g_c is a constant to be determined and G_0, g_0 are arbitrary constants.

Evidently h will have the same form as g, but its non-periodic part will be $h_c\tau + h_0$.

When these results are substituted in the second term of R to obtain the portion factored by k_1, any term in this second portion which contains the argument h will have the argument $h_c\tau + h_0$, and consequently can produce no new constant parts in g or h to this order. This justifies the neglect of terms with the argument $2\varpi' + 2h$ in the calculation of $(R_f, S)_c$ performed in § 7. We therefore have h absent from both parts if R and H remain constant.

Since the mean motion of f' is n', the actual mean motions will be

$$\tfrac{3}{8}\lambda\left(\frac{n'}{n}\right)^2 ng_c, \qquad \tfrac{3}{8}\lambda\left(\frac{n'}{n}\right)^2 nh_c, \qquad \lambda = \frac{m_3(1-e'^2)^{-\frac{3}{2}}}{m_2+m_2+m_3}.$$

Since the factor L ultimately disappears from these mean motions I shall abbreviate the exposition by putting $L = 1$, so that now

$$G^2 = 1 - e^2, \qquad H^2 = \cos^2 i(1-e^2) = (1-s^2)(1-e^2).$$

10. We can now write

$$R = -A + B\cos 2g, \tag{10·1}$$

where A, B are functions of G and the equations are

$$\frac{dG}{d\tau} = -2B\sin 2g, \qquad \frac{dg}{d\tau} = \frac{\partial A}{\partial G} - \frac{\partial B}{\partial G}\cos 2g.$$

Since τ is not present in R these equations have an integral

$$A - B\cos 2g = \text{const.} = A_0, \tag{10·2}$$

where A_0 is the value of A when $G = G_0$, $2g = 90°$.

We have

$$d\tau = \frac{dG}{\{B^2 - (A-A_0)^2\}^{\frac{1}{2}}},$$

which reduces the solution to a quadrature. It is easily seen from the expression for R in § 8 that this can be written

$$d\tau = \frac{dG^2}{(\text{cubic in. } G^2)^{\frac{1}{2}}},$$

which is an elliptic integral.

When the numercial values of G_0, H are given, a more convenient method is to obtain G from (10·2) in the form

$$G = \Sigma G_i' \cos 2ig$$

by harmonic analysis. This is easily done, because for each value of $2g$ the solution of a quadratic gives G^2 and therefore G. From this we compute the Fourier series for $1 \div \partial R/\partial G$, and the solution of the equation

$$\tau = -\int dg \div \partial R/\partial G$$

immediately follows. A reversion of series then gives g in terms of τ.

Neither of these methods appeared to be convenient for a lateral development. It was found to be possible to obtain such a development in powers of $B_0 \div \partial A_0/\partial G_0$. As B_0 has the factor $e^2 \sin^2 i$ and the principal part of $\partial A_0/\partial G_0$ is 4, such a plan is well suited to the needs of the astronomical problem.

11. Put $G = G_0 + \delta G$ and expand (10·2) in powers of δG. We obtain

$$\frac{\partial A_0}{\partial G_0}\delta G + \frac{1}{2}\frac{\partial^2 A_0}{\partial G_0^2}(\delta G)^2 + \ldots - \left(B_0 + \frac{\partial B_0}{\partial G_0}\delta G + \ldots\right)\cos 2g = 0.$$

Denote derivatives with respect to G_0 by suffixes. Then, if we neglect quantities of the order B_0^2/A_1^2, we have

$$\delta G = \frac{B_0 \cos 2g}{A_1 - B_1 \cos 2g};$$

and to the same degree of accuracy,

$$\frac{dg}{d\tau} = A_1 - B_1 \cos 2g + (A_2 - B_2 \cos 2g)\delta G.$$

Define the angle ψ by means of the relation

$$\tan \psi = \left(\frac{A_1 + B_1}{A_1 - B_1}\right)^{\frac{1}{2}} \tan g,$$

a relation which is familiar as being the same as that in an elliptic orbit between the true and eccentric anomalies. This equation gives

$$(A_1 - B_1 \cos 2g)(A_1 + B_1 \cos 2\psi) = A_1^2 - B_1^2 = C^2,$$

$$\cos 2g = \frac{A_1 \cos 2\psi + B_1}{A_1 + B_1 \cos 2\psi}, \qquad \delta G = B_0 \frac{A_1 \cos 2\psi + B_1}{C^2},$$

$$\frac{dg}{d\tau} = \frac{d\psi}{d\tau} \cdot \frac{A_1 - B_1 \cos 2g}{C} = \frac{C}{A_1 + B_1 \cos 2\psi}.$$

If we neglect B_0 we have $d\psi/d\tau = (A_1^2 - B_1^2)^{\frac{1}{2}} = C$, so that in the expression for δG which has the factor B_0, $\cos \psi$ has a zero mean value. The usual definition of the "eccentricity" in disturbed motion requires that the coefficient of the principal elliptic term shall have the same form as in undisturbed motion. The constant part of G must therefore be G_0 when e becomes e_0. We must therefore define δG by

$$\delta G = B_0 \frac{A_1 \cos 2\psi}{A_1^2 - B_1^2} = \frac{B_0 A_1}{C^2} \cos 2\psi$$

instead of the value given above.

On replacing the variable g by ψ in the equation for g we obtain

$$\frac{d\psi}{dt} = C + \frac{B_0 A_1 \cos 2\psi}{C^3}\{A_2(A_1 + B_1 \cos 2\psi) - B_2(B_1 + A_1 \cos 2\psi)\}.$$

Putting $d\psi/dt = C$ in the terms factored by B_0 and integrating, we obtain a result which may be put into the form,

$$\psi = \tau\left\{C - \tfrac{1}{2}B_0\frac{d}{dG_0}\frac{B_1}{C}\right\} - \tfrac{1}{4}B_0A_1\frac{d}{dG_0}\frac{1}{C}\sin 2\psi_0 - \tfrac{1}{16}B_0\frac{d}{dG_0}\left(\frac{A_1^2}{C^2}\right)\sin 4\psi_0,$$

where ψ_0 is the non-periodic part of ψ.

The mean motion g_m of g is the same as that of ψ. Whence, remembering that

$$C^2 \equiv A_1^2 - B_1^2,$$

we obtain

$$g_m = \tfrac{3}{8}\lambda\left(\frac{n'}{n}\right)^2 n\left(C - \tfrac{1}{2}B_0\frac{d}{dG_0}\frac{B_1}{C}\right).$$

12. The determination of h_m is more troublesome. We have

$$\frac{dg}{d\tau} = A_1 - B_1\cos 2g + \delta G(A_2 - B_2\cos 2g),$$

$$\frac{dh}{d\tau} = A' - B'\cos 2g + \delta G(A_1' - B_1'\cos 2g),$$

where accents denote derivatives with respect to H. Hence

$$\frac{dh}{d\tau} = A' - \frac{B'}{B_1}A_1 + \frac{B'}{B_1}\frac{dg}{d\tau} + \delta G\left\{A_1' - \frac{B'}{B_1}A_2 - \left(B_1' - \frac{B'}{B_1}B_2\right)\cos 2g\right\}.$$

But

$$\delta G\cos 2g = B_1\frac{A_1\cos 2\psi}{C^2}\cdot\frac{A_1\cos 2\psi + B_1}{A_1 + B_1\cos 2\psi},$$

$$= \frac{B_0A_1}{B_1^2}\left(\frac{A_1B_1}{C^2}\cos 2\psi - 1 + \frac{A_1}{A_1 + B_1\cos 2\psi}\right).$$

Since we can put $d\psi/d\tau = C$ in the terms factored by B_0, the equation for h is now integrable. Remembering that the mean value of $(A_1 + B_1\cos 2\psi)^{-1}$ is C^{-1}, we obtain for that of $\delta G\cos 2g$

$$\frac{B_0A_1}{B_1^2}\left(\frac{A_1}{C} - 1\right) = \frac{B_0A_1}{C(A_1 + C)}.$$

The mean value of $dh/d\tau$ is therefore

$$A' + \frac{B'}{B_1}(C - A_1) - \tfrac{1}{2}B_0\frac{B'}{B_1}\frac{d}{dG_0}\frac{B_1}{C} - \frac{B_0B_1A_1}{C(A_1 + C)}\frac{d}{dG_0}\frac{B'}{B_1}.$$

The third of these terms can be written

$$-\tfrac{1}{2}B_0\left(\frac{d}{dG_0}\frac{B'}{C} - \frac{B_1}{C}\frac{d}{dG_0}\frac{B'}{B_1}\right),$$

and the second with the latter of these two terms gives

$$-\frac{B_0B_1}{C}\left(\frac{A_1}{A_1 + C} - \frac{1}{2}\right)\frac{d}{dG_0}\left(\frac{B'}{B_1}\right) = -\frac{1}{2}\frac{B_0B_1}{C}\cdot\frac{B_1'B_1 - B'B_2}{(A_1 + C)^2}.$$

One obvious transformation frees the second term from the divisor B_1, which may be very small. We finally obtain

$$h_m = \tfrac{3}{8}\lambda\left(\frac{n'}{n}\right)^2 n\left\{A' - \frac{B'B_1}{A_1+C} - \frac{1}{2}\frac{d}{dG}\frac{B'}{C} - \frac{1}{2}\frac{B_0B_1(A_1'B_1 - B'B_2)}{C(A_1+C)^2}\right\}.$$

The periodic part of h can be found in terms of ψ. The method requires that $|B_1| < |A_1|$, or since B_1, A_1 are positive, $B_1 < A_1$. More generally, the equation

$$\frac{dg}{d\tau} = \frac{\partial A}{\partial G} - \frac{\partial B}{\partial G}\cos 2g$$

shows that if $|dB/dG| > |dA/dG|$ the angle g will oscillate about a mean value zero, so that the apse is retrograding with the same mean velocity as the node. There are interesting details concerning the cases where the inclination is large which I shall leave to a later occasion.

13. The functions B_0, B_1, B' respectively have e^2s^2, $e^2 - s^2$, e^2 as factors, where $s \equiv \sin i$. For many purposes, and especially for the lunar theory, it is useful to expand in powers of e^2, s^2. Since we have neglected B_0^2 the errors of the formulæ are of the eighth order with respect to e, s.

If we expand in powers of B_1, B', it is easily seen that, to the sixth order inclusive,

$$g_m = \frac{3}{8}\lambda\left(\frac{n'}{n}\right)^2 n\left(A_1 - \frac{1}{2}\frac{d}{dG_0}\frac{B_0B_1}{A_1}\right), \qquad h_m = \frac{3}{8}\lambda\left(\frac{n'}{n}\right)^2 n\left(A' - \frac{1}{2}\frac{d}{dG_0}\frac{B_0B'}{A_1}\right).$$

Referring back to the expression for R in § 8, we have

$$A_1 = -G_0 + \frac{5H^2}{G_0^3} + Hk_1\left(33G_0 + 15\frac{H^2}{G_0^3}\right),$$

$$A' = \left(3 - \frac{5}{G_0^2}\right)H + \tfrac{1}{2}k_1\left(-35 + 33G_0^2 - 45\frac{H^2}{G_0^2} + 51H^2\right),$$

$$B_0 = \tfrac{5}{2}(1 + 3Hk_1)(1 - G_0^2)\left(1 - \frac{H^2}{G_0^2}\right), \text{ etc.},$$

where

$$k_1 = \tfrac{3}{8}(1 + \tfrac{2}{3}e'^2)(1 - e'^2)^{-\frac{3}{2}}\lambda\frac{n'}{n}.$$

These derivatives are expanded in powers of e_0^2, $s_0^2 = \sin^2 i_0$ with $G_0^2 = 1 - e_0^2$, $H^2 = (1 - s_0^2)(1 - e_0^2)$. In giving the results I omit the suffix of e_0 and put $s_0^2 = 4\gamma^2 - 4\gamma^4$ in accordance with the usual practice. Also h_m and g_m are added together to give ϖ_m, the mean motion of the apse. The results are if, for brevity, we put

$$\lambda_1 = \frac{3}{8}\frac{m_3}{m_1+m_2+m_3}\left(\frac{n'}{n}\right)^2(1 - e'^2)^{-\frac{3}{2}}n,$$

$$\lambda_2 = \frac{9}{64}\left(\frac{m_3}{m_1+m_2+m_3}\right)^2\left(\frac{n'}{n}\right)^3(1 + \tfrac{2}{3}e'^2)(1 - e'^2)^{-3}n,$$

$$\varpi_m = \lambda_1(2 - e^2 - 16\gamma^2 - \tfrac{1}{4}e^4 + 23e^2\gamma^2 - 30\gamma^4 - \tfrac{1}{4}e^6 - \tfrac{37}{4}e^4\gamma^2 + \tfrac{345}{2}e^2\gamma^4 - 150\gamma^6)$$
$$+ \lambda_2(50 - 75e^2 - 168\gamma^2 + 0e^4 + 18e^2\gamma^2 + 492\gamma^4 + 0e^6 + \tfrac{1425}{4}e^4\gamma^2$$
$$- \tfrac{7179}{2}e^2\gamma^4 + 2430\gamma^6),$$

$$h_m = \lambda_1(-2 - 4e^2 + 4\gamma^2 + \tfrac{7}{8}e^4 - 17e^2\gamma^2 + 0\gamma^4 - \tfrac{31}{32}e^6 + \tfrac{261}{8}e^4\gamma^2 - 50e^2\gamma^4 + 0\gamma^6)$$
$$+ \lambda_2(2 - 42e^2 - 12\gamma^2 - \tfrac{75}{4}e^4 + 252e^2\gamma^2 + 12\gamma^4 + \tfrac{1275}{32}e^6 - \tfrac{7575}{8}e^4\gamma^2$$
$$+ 1173e^2\gamma^4 + 0\gamma^6).$$

Delaunay has given * these terms as far as the fourth order in e, e', γ with the exception of that with factor $m^3e'^4$, and they agree with mine after the correction of an error in Delaunay's work, which I detected long ago.† On making use of the expansion of the factors dependent on e' given in § 7, we find that the terms dependent on e'^4 are

$$(\tfrac{3}{4} \cdot \tfrac{15}{8} + 8 \cdot \tfrac{225}{32}m)m^2e'^4n = \tfrac{45}{32}(1 + 40m)m^2e'^4n.$$

As $m = \cdot 075$, the second of these terms contributes three times as much as the first, which adds $1''$ to the centennial motion of the perigee. My estimate for the whole contribution of the terms factored by e'^4 is $11'' \pm 2''$, this estimate being aided by the terms dependent on e'^2 which Delaunay gives as far as m^5 and for which I found the complete numerical value. (I tried an estimation of these before looking them up and found a factor 7·0 instead of 7·5 for the term $m^4e'^2$ and 12·0 instead of 13·3 for the term $m^5e'^2$.) The terms dependent on e'^4 were not calculated in my theory.

The sixth-order terms give the following additions to the centennial motions :—

$$\varpi_m,\ 0 - 0''\cdot 6(1 - 1\cdot 05) + 7''\cdot 6(1 - \cdot 58) - 4''\cdot 5(1 - \cdot 44)\ ;$$
$$h_m,\quad -0''\cdot 1(1 - 1\cdot 15) + 8''\cdot 7(1 - \cdot 80) - 2''\cdot 2(1 - \cdot 65) + 0,$$

where the four portions are the terms factored by e^6, $e^4\gamma^2$, $e^2\gamma^4$, γ^6 and the two in the brackets the parts dependent on m^2, m^3. From these I estimate the addition of $5'' - 2'' = 3''$ to the motion of the perigee and $-5'' + 1'' - 4''$ to that of the node.

In my last comparison ‡ between theory and observation for these motions I found that $17''$ should be added to the theoretical value for agreement. The additions given above account for $14''$, and, if we add the relativity portion $2''$ obtained by de Sitter, § the whole difference is accounted for. I estimate the probable error of the comparison as $\pm 5''$.

For the node the difference was $-12''$ and the additions make it $-16''$. This is less disturbing, because the motion of the node is not so well determined from observation as that of the perigee, and because the portion added is due to the figure of the Moon, $-14''$ may be in error by its whole amount. This latter portion shows the necessity for a new determination of the moments of inertia of the Moon.

In the new numerical verification now under way by Professor W. J. Eckert and myself it is hoped that the solar portions of these centennial motions may be obtained accurately to $1''$.

* *C.R.*, 74, 18, 1872.
† *M.N.*, 57, 341, 1897.
‡ *M.N.*, 74, 419, 1914.
§ *M.N.*, 77, 172, 1916.

14. The apse node terms in the mean longitude are obtained from

$$\frac{d}{d\tau}(w - nt - e) = -\frac{\partial R}{\partial L} - \frac{\partial R}{\partial G} - \frac{\partial R}{\partial H}.$$

But

$$L\frac{\partial R}{\partial L} + G\frac{\partial R}{\partial G} + H\frac{\partial R}{\partial H} = 4R = \text{const.}$$

Hence, omitting the constant we have when $L = 1$,

$$\frac{d}{d\tau}\delta w = (G-1)\frac{\partial R}{\partial G} + (H-1)\frac{\partial R}{\partial H} = (1-G)\frac{dg}{d\tau} + (1-H)\frac{dh}{d\tau},$$

which is easily integrated with the known values of G, g, h.

Yale Observatory :
1936 *November* 16.

POLYTROPIC INDEX AND PHOTOSPHERIC CONDITIONS.

Henry Norris Russell.

Professor Milne's paper "The Pressure in the Photosphere of a Star" * presents two problems which may be stated as follows :—

Suppose that inside a spherical star a sphere (1) of radius r_1 encloses the mass M_1, and that, on this sphere, the pressure is P_1 and the density ρ_1. Suppose also that

(*a*) The relations connecting P_1, ρ_1, M_1 and r_1 are identical with those in a polytropic sphere of index n, without central singularity.

(*b*) The material at r_1 is a perfect gas of molecular weight μ_1 and temperature T_1.

The problems are :

(1) Given M_1, r_1, μ_1, T_1, n ; to find P_1.
(2) Given M_1, r_1, μ_1, T_1, P_1 ; to find n.

Milne gives a simple solution for (1) and suggests (p. 203) a rather complicated graphical procedure for (2), remarking "How practicable is this method, I cannot say."

It may be simply solved as follows : Let $\bar{\rho}_1$ be the mean density of sphere (1) and $\bar{P}_1$ the central pressure if this sphere were homogeneous.

Then

$$\bar{\rho}_1 = \frac{3}{4\pi}\frac{M_1}{r_1^3}, \qquad \bar{P}_1 = \frac{3G}{8\pi}\frac{M_1^2}{r_1^4}. \tag{1}$$

At any point we may set †

$$\rho = X\bar{\rho}_1, \qquad P = Y\bar{P}_1, \tag{2}$$

* *M.N.*, **96**, 195, 1936. † Russell, Dugan and Stewart, *Astronomy*, **2**, 896.

where X, Y are functions of r. For a polytropic sphere defined by Emden's equation $\frac{d^2y}{dx^2}+\frac{2}{x}\frac{dy}{dx}+y^n=0$,

$$X=-\frac{x_1}{3y_1'}y^n, \qquad Y=\frac{2}{3(n+1)y_1'^2}y^{n+1}, \tag{3}$$

where x is proportional to r, $y=1$ at the centre and $y'=dy/dx$. If the gas-pressure at r_1 is β_1 times the total pressure,

$$p_1=\beta_1 P_1=R\rho_1 T_1/\mu_1.$$

Then

$$\frac{P_1}{\rho_1}=\frac{RT_1}{\beta_1\mu_1}=\frac{\bar{P}_1}{\bar{\rho}_1}\cdot\frac{Y_1}{X_1}=\frac{G}{2}\frac{M_1}{r_1}\cdot\frac{-2y_1}{(n+1)x_1y_1'}. \tag{4}$$

If $\bar{T}_1$ is the gas-temperature corresponding to $\bar{P}_1$, $\bar{\rho}_1$, μ_1,

$$\bar{T}_1=\frac{\mu_1 G}{2R}\frac{M_1}{r_1}, \tag{5}$$

and (4) may be written

$$\frac{T_1}{\beta\bar{T}_1}=-\frac{2y_1}{(n+1)x_1y_1'}. \tag{6}$$

As y, y' are known functions of x (for a given n) this determines x_1.

With the Sun's mass and radius as units,

$$\bar{T}_1=1{\cdot}15\times10^7\mu_1 M/r. \tag{7}$$

If, following Milne, we take T_1 equal to the effective photospheric temperature, y_1/x_1y_1' will be less than 0·001 for most stars and less than 0·01 even for super-giants.

Now, for all values of n between 1 and 5, $-y/xy'$ is a monotonically decreasing function of x,* which falls to zero, with finite slope, at the boundary $x=x_0$. There is always one, and only one, real solution of (6). With data representing actual stars, y_1 is very small.

We have then, neglecting terms of the second order,

$$x_0-x_1=-y_1/y_1'=\frac{n+1}{2}x_1\frac{T_1}{\beta_1\bar{T}_1}, \tag{8}$$

a remarkably simple equation. The outer radius, r_0, of the polytropic sphere therefore exceeds r_1 very little (usually by less than one part in a thousand). The total mass M_0 usually exceeds M_1 by less than one part in a million.

Within the errors of the best observations we may therefore set the boundary values x_0, y_0' in place of x_1, y_1', and obtain

$$y_1=-\frac{(n+1)x_0y_0'}{2}\cdot\frac{T_1}{\beta_1\bar{T}_1}=F_1(n)\frac{T_1}{\beta_1\bar{T}_1}. \tag{9}$$

* British Association Mathematical Tables, Vol. II, *Emden Functions*, London, 1932. The tabular quantity $V=-\frac{(n+1)xy'}{y}$.

Equation (4) may also be written

$$r_1 = \frac{\beta_1 G M_1}{(R/\mu_1) T_1} \cdot \frac{1}{n+1} \cdot \frac{y_1}{-x_1 y_1'}. \tag{10}$$

Substituting in (1) we find, using (2) and (3),

$$p_1 = \frac{(R/\mu)^4 T_1^4}{4\pi \beta_1^3 G^3 M_1^2} (n+1)^3 y_1^{n-3} (-x_1^2 y_1')^2. \tag{11}$$

These are identical with Milne's equations (10) and (9) (pp. 198–199). The simple equations (8) and (9) (which appear to be new) facilitate their application.

When $p_1 T_1$ are given, and n is required, we have, by (1), (2), (3) and (6),

$$\frac{p_1}{\bar{P}_1} = \frac{(n+1)^n}{3 \cdot 2^n} x_1^{n+1} (-y_1')^{n-1} \beta_1^{-n} \left(\frac{T_1}{\bar{T}_1}\right)^{n+1}. \tag{12}$$

As before, we may set $x_1 = x_0$.

If

$$\frac{(n+1)^n}{3 \cdot 2^n} x_0^{n+1} (-y_0')^{n-1} = F_2(n), \tag{13}$$

we have, with abundant approximation,

$$(n+1) \log (T_1/\bar{T}_1) = \log (p_1/\bar{P}_1) - \log F_2(n) + n \log \beta_1. \tag{14}$$

The values of $F_1(n)$ and $\log_{10} F_2(n)$ are as follows :—

n	$F_1(n)$	$\log_{10} F_2(n)$	n	$F_1(n)$	$\log_{10} F_2(n)$
0·0	1·000	0·00	3·0	0·585	1·04
0·5	1·032	0·27	3·5	0·446	0·96
1·0	1·000	0·52	4·0	0·300	0·70
1·5	0·929	0·73	4·5	0·150	0·08
2·0	0·832	0·90	4·9	0·030	−1·50
2·5	0·715	1·01	5·0	0·000	$-\infty$

They are taken from the British Association Tables, except for $n = 0{\cdot}5$ and 4·9, where only Emden's older values are available. If m is taken as vertical and n as horizontal co-ordinate, the solution of (14) is given by the intersection of the fixed curve $m = -\log F_2(n)$ with the straight line

$$m = n \log (T_1/\beta_1 \bar{T}_1) - \log (p_1/\bar{P}_1) + \log (T_1/\bar{T}_1).$$

A plot shows that there will be one and only one real solution, with n lying between 0 and 5, so long as $p_1/\bar{P}_1 < T_1/\bar{T}_1$—that is, so long as $\rho_1 < \bar{\rho}_1$. But this is a necessary, though by no means a sufficient, condition that the approximations here employed shall hold good. Hence, with data derived from actual stars, there is always one real solution.

For most stars we may follow Milne and set $\beta_1 = 1$. Equation (14) may then be easily solved by successive approximations, starting with

$$\log F_2(n) = -0{\cdot}9$$

(the mean value from $n = 1{\cdot}5$ to $n = 4{\cdot}0$).

For the Sun, $\bar{T}_1 = 1{\cdot}15 \times 10^7 \mu_1$, $\bar{P}_1 = 1{\cdot}35 \times 10^{15}$ dyn./cm.^{2}. With the values adopted by Milne—$T_1 = 5740°$, $\mu_1 = 1$, $p_1 = 184$ dyn./cm.^{2},* $\beta_1 = 1$—we have $\log (T_1/\bar{T}_1) = -3{\cdot}30$, $\log (p_1/\bar{P}_1) = -12{\cdot}87$. Starting with $\log F_2 = -0{\cdot}9$, we find $n = 3{\cdot}15$; 2nd approximation, $\log F_2 = -1{\cdot}04$, $n = 3{\cdot}22$; 3rd approximation, $\log F_2 = -1{\cdot}03$, $n = 3{\cdot}22$—in substantial agreement with Milne's estimate "very slightly greater than 3."

Hence, by (8), $x_1 - x_0 = 0{\cdot}0021\ x_0$; by (9), $y_1 = 0{\cdot}00028$.

The adopted value of p_1 is derived on the assumption that the whole atmosphere consists of a single active constituent, which is half ionized, so that the electron pressure $p_E = \frac{1}{3} p_1 = 61$ dyn./cm.^{2}.

Now the effective electron pressure "at the photosphere" of the Sun appears to be fairly well determined, but the total pressure is less certain. For example, the writer,† by a method differing considerably from Chandrasekhar's, though using the same data, finds $p_E = 46 \pm 9$ dyn./cm.^{2}; but his final conclusion ‡ was that hydrogen atoms are about 300 times as abundant as electrons and 30 times as abundant as any other constituent, making p_1 about 15000 dyn./cm.^{2}. We now have $\log (p_1/\bar{P}_1) = -10{\cdot}95$, whence $n = 2{\cdot}63$. Reduction of the hydrogen abundance and of p_1 to one-tenth of this amount gives $n = 2{\cdot}94$—so that n is not very sensitive to the assumed pressure.

For stars of various types, indeed, the values of n come out a good deal the same. The data adopted by the writer § give a good enough account of the behaviour of spectral lines and bands to be accepted as fairly representative. Taking the values for the Sun as units, and setting $z' = 5040/T_1$, these give:

	Giants	Main Sequence
Surface gravity	$\log g_1 = 0{\cdot}76 - 2{\cdot}23\ z'$	$-0{\cdot}65 + 0{\cdot}64\ z'$
Number of atoms above 1 cm.^{2} of photosphere	$\log N_1 = 0{\cdot}20 + 0{\cdot}35\ z'$	$0{\cdot}40 - 0{\cdot}45\ z'$
Photospheric pressure $p_1 = Ng$	$\log p_1 = 0{\cdot}96 - 1{\cdot}88\ z'$	$-0{\cdot}25 + 0{\cdot}19\ z'$
$\bar{P}_1 = \frac{3}{8\pi G} g_1^2$	$\log \bar{P}_1 = 1{\cdot}52 - 4{\cdot}46\ z'$	$-1{\cdot}30 + 1{\cdot}28\ z'$
In dynes/cm.^{2} we have	$\log p_1 = 5{\cdot}14 - 1{\cdot}88\ z'$	$3{\cdot}93 + 0{\cdot}19\ z'$
	$\log \bar{P}_1 = 16{\cdot}65 - 4{\cdot}46\ z'$	$13{\cdot}83 + 1{\cdot}28\ z'$

From the observational data for the giants,‖ we have empirically in solar units from $z' = 0{\cdot}6$ to $1{\cdot}7$, $\log M_1/r_1 = 0{\cdot}45 - 0{\cdot}90\ z'$, whence in degrees K, $\log \bar{T}_1 = 7{\cdot}51 - 0{\cdot}90\ z' + \log \mu_1$. For the main sequence, the observed values of M_1/r_1 give (in solar units):

z'	0·3	0·4	0·5	0·6	0·8	1·0	1·2	1·4
$\log \frac{M}{r}$	0·18	0·06	0·02	0·00	−0·02	−0·03	−0·04	−0·05

* Chandrasekhar, *M.N.*, **92**, 186, 1932.
† *Ap. J.*, **78**, 277, 1933.
‡ *Ap. J.*, **79**, 323–324, 1934.
§ *Ap. J.*, **79**, 323, 327, 1934.
‖ *Ap. J.*, **78**, 271, 1933.

Using these values, and setting $\mu_1 = \beta_1 = 1$, we have, for giant stars :

TABLE I

z	T_1	$\log p_1$	$\log \bar{P}_1$	$\log T_1$	$\log \bar{T}_1$	$\log (p_1/\bar{P}_1)$	$\log (T_1/\bar{T}_1)$	n
0·8	6300°	3·64	13·08	3·80	6·79	−9·44	−2·99	2·49
1·0	5040	3·26	12·19	3·70	6·61	−8·93	−2·91	2·40
1·2	4200	2·88	11·29	3·62	6·43	−8·41	−2·81	2·34
1·4	3600	2·50	10·40	3·56	6·25	−7·90	−2·69	2·30
1·6	3150	2·13	9·50	3·50	6·07	−7·37	−2·57	2·24

For the main sequence, we may use the formulæ given above up to $T_1 = 6300°$. Above 10000°, where the hydrogen is almost completely ionized, we must set $\mu_1 = 0·5$, and diminish N_1 and p_1 by a factor of about 100, on account of the great increase in opacity. For $T_1 = 8400°$ the transition is well advanced, and we may divide p by 10 and set $\mu_1 = 0·55$. We thus obtain :

TABLE II

z	T_1	$\log p_1$	$\log \bar{P}_1$	$\log T_1$	$\log \bar{T}_1$	$\log (p_1/\bar{P}_1)$	$\log (T_1/\bar{T}_1)$	n
0·3	16800°	1·99	14·21	4·23	6·94	−12·22	−2·71	3·80
0·4	12600	2·01	14·34	4·10	6·82	−12·33	−2·72	3·81
0·5	10080	2·03	14·47	4·00	6·78	−12·44	−2·78	3·76
0·6	8400	3·04	14·60	3·92	6·80	−11·56	−2·88	3·37
0·8	6300	4·08	14·85	3·80	7·04	−10·77	−3·24	2·64
1·0	5040	4·12	15·11	3·70	7·03	−10·99	−3·33	2·62
1·2	4200	4·16	15·37	3·62	7·02	−11·21	−3·40	2·60
1·4	3600	4·20	15·62	3·56	7·01	−11·42	−3·45	2·60

With the high abundance of hydrogen here assumed, n is near 2·5 for all the cooler stars—giants and dwarfs. Of the increase of n for the hot stars, about one-third arises from the division of μ_1 by 2, and two-thirds from that of p_1 by 100. Diminishing the hydrogen abundance by a factor of 10 would increase n by about 0·3 for the cooler stars, leaving the hotter ones unchanged.

Analysis on Milne's principles therefore leads, for stars of all the important types, to values of n not very different from that of the "standard model." The significance of this result deserves consideration.

It should be emphasized that the only properties of the star as a whole which have been used are its mass and radius. The values of p_1, T_1, μ_1 refer to the photospheric layer alone, while the auxiliary quantities $\bar{P}_1$, $\bar{\rho}_1$, $\bar{T}_1$ are convenient functions of M_1, r_1, μ_1, having the dimensions of pressure, density and temperature. No assumption has been made at any point in the argument that the pressure, temperature, etc. in the deep interior had these values (though it is easy to show that each of these will almost certainly be reached at some internal point).

The analysis shows that there is one, and only one, value of n such

that for a star of the given M_1 and r_1, and the given photospheric p_1, T_1, μ_1 (and hence ρ_1), a solution of Emden's equation, carried inward to the centre, will meet with no singularity there; and the computations show that this value of n always lies between 2 and 4. But it does not prove that the internal constitution of the star actually follows this polytropic model. There must be a vast variety of solutions, which give the same mass and radius, and the same pressure at the level where the temperature is T_1.

Roughly speaking, we may say that the Emden solution, with the determined index n, indicates the order of magnitude of the central condensation —or, if we will, the mean compressibility; but this is only true in a very general and approximate sense.

The similarity indicated by the general rough agreement of the values of n for stars of different types is more apparent than real. One of the most important physical properties of the stars is that the effective temperatures of their surfaces are in all cases of the order of magnitude of a thousandth part of the temperatures in the deep interior (which are always of the order of magnitude of T_1). It follows at once, from (14), that to change n by a unit changes p_1 by a factor of the order of 1000. For a star of given mass, size and effective temperature, the tabular range in n therefore corresponds to a change in the photospheric pressure by more than ten thousand fold. When more accurate values of this pressure become available, the computed value of n will afford a valuable parameter for description of certain properties of the photosphere—or, rather, of the relation of these properties to those of the star as a whole; but it will be less useful as an index of internal constitution.

Princeton University Observatory:
1936 *November* 10.

THE OCCURRENCE OF NEGATIVE DENSITY GRADIENTS IN STARS AND ALLIED PROBLEMS. (FIRST PAPER.)

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1. The question as to how far the distribution of the energy sources affects the possible *types* of density distributions that can occur in the interior of a star is one concerning which we have at present very little information. That the problem is one which needs careful consideration has been pointed out by Rosseland,* who found that even with a mild concentration of the energy sources towards the centre one encounters the occurrence of *negative density gradients—i.e.* an initial rise in density on going outwards from the centre of the star. Rosseland further points out that "stars with a density

* S. Rosseland, *Zeits. f. Astrophysik*, 4, 255, 1932. Referred to as *loc. cit.* Also G. Steensholt, *ibid.*, 5, 140, 1932.

distribution of this type are scarcely likely to remain in equilibrium without suffering collapses now and again, during which their inner configuration is upset, *or* without having a permanent system of convection currents set up. It may thus happen that *all* stars have an extended core which is *convectively unstable*" * (*loc. cit.*). However, the particular configuration for which Rosseland gives numerical figures corresponds to a "massive" star with a central radiation pressure $(1-\beta_c)=0{\cdot}67$. One can perhaps conjecture on the basis of Rosseland's figures that the occurrence of negative density gradients is possibly a characteristic of only the massive stars with large central radiation pressures.† More precisely, the conjecture is that for a given degree of the concentration of the energy sources towards the centre it should be possible to set a lower limit to the central radiation pressure such that a configuration with a higher value for the central radiation pressure would be characterized by a negative density gradient in the neighbourhood of the centre. The object of this paper is to see in how far the above conjecture can be substantiated. In the analysis we shall confine ourselves to a certain special class of model source distributions. For these models (more precisely described in § 2) the integration of the equations of equilibrium can be effected explicitly to such an extent as to enable an analytical (as distinct from a numerical) discussion of the different *types* of density distributions that *can* occur and the circumstances under which they may be expected to occur.

2. In this paper we shall in the first instance confine ourselves to stellar models, in which

$$\eta=\frac{L(r)}{M(r)}\cdot\frac{M}{L}\propto T^{\delta},\qquad(\delta>0)\qquad(1)\ddagger$$

where T is the temperature and the other symbols have their usual meanings. Later we shall extend the analysis to the more general case where

$$\eta\propto\rho^{\alpha}T^{\delta},\qquad(2)$$

where ρ is the density. Further, for the law of opacity we shall assume

$$\kappa=\kappa_1\frac{\rho}{T^{3\cdot5}}.\qquad(3)$$

The formal analysis can easily be extended to include more general laws of opacity. The particular class of source distributions in which we shall be most interested are those for which

$$\eta\propto\rho^{\alpha}T^{4\cdot5+\alpha}.\qquad(4)$$

For the law (4) the equations of equilibrium can be integrated sufficiently to enable one to specify the circumstances under which negative density gradients will occur.

* The italics are ours.

† By "central radiation pressure" we mean the ratio $\frac{1}{3}aT_c^4/P_c=1-\beta_c$.

‡ An assumption of this kind was first used by Jeans, but in his work he almost always put $\delta=0$, and once considered small *negative values* for δ. But we shall be concerned with $\delta>0$.

That the particular distribution of the energy sources assumed in (1) or (2) cannot occur in Nature is of course clear. Further, with the usual boundary condition used for stellar models, namely, that $T=0$ at the boundary,* leads to $\eta=0$ at the boundary, while of course $\eta \to 1$ as $r \to R$, the radius of the star. This last, however, is not a serious difficulty. We can write (1) as

$$\eta = \left(\frac{T}{T^*}\right)^{\delta}, \tag{5}$$

where T^* *is the temperature at a point well inside the star.* When $T=T^*$, $\eta=1$, and we shall then have to assume that, further out, η remains constant and equal to unity—*i.e.* we should have to consider the star as having a "point-source" envelope outside $T=T^*$.

The main point, however, in studying models based on the assumption (1) or (2) is simply that then the equations of equilibrium can be discussed analytically. Further, (1) and (2) both provide methods—very special though they may be—of concentrating the energy sources towards the centre, and consequently one may at least expect the general features to hold good *qualitatively* under less specialized circumstances.

3. *Equations of the Problem.*—We start as usual with the equations

$$\frac{d(p+p_r)}{dr} = -G\frac{M(r)}{r^2}\rho, \tag{6}$$

$$\frac{dp_r}{dr} = -\frac{\kappa L(r)}{4\pi c r^2}\rho, \tag{7}$$

where p and p_r denote the gas and the radiation pressures respectively. With the law of opacity given by (3), we derive from (6) and (7) that

$$\frac{d}{dp_r}\left[p_r\left(1+\frac{p}{p_r}\right)\right] = \frac{4\pi cGM}{\kappa_1 L}\left(\frac{M(r)}{M}\cdot\frac{L}{L(r)}\right)\frac{T^{3.5}}{\rho}. \tag{8}$$

Introduce the variable y defined by

$$y = \frac{p}{p_r} = \frac{k}{\mu H}\frac{3}{a}\frac{\rho}{T^3}, \tag{9}$$

where k is the Boltzmann constant, μ the molecular weight, H the mass of the proton and a the radiation constant. Further, we shall assume that

$$\eta = \eta_0 T^{\delta}. \tag{10}$$

(8) can now be rewritten in the form

$$p_r\frac{dy}{dp_r} = K\frac{T^{(\frac{1}{2}-\delta)}}{y} - (1+y), \tag{11}$$

* The "difficulty" is not removed by using the boundary condition advocated by Milne, namely, that $T \to T_0$ (the boundary temperature), as $\eta \to 1$. This is clear when we write η in the form (5). If we took the law for η too literally, then we should identify T^* with T_0, and the difficulties which would be created by this are obvious—we would be creating energy *sinks* in the main body of the star. In discussing models of the type we are considering one should exercise great caution.

where

$$K=\frac{4\pi cGM}{\kappa_1\eta_0 L}\frac{3}{a}\frac{k}{\mu H}. \tag{12}$$

Introduce the new variable x defined by

$$x=KT^{(\frac{1}{2}-\delta)}. \tag{13}$$

(11) takes the form

$$\gamma xy\frac{dy}{dx}=y(y+1)-x, \tag{14}$$

where

$$\gamma=\tfrac{1}{4}(\delta-\tfrac{1}{2}), \tag{15}$$

and

$$x=KT^{-4\gamma}; \qquad y=\frac{\beta}{1-\beta}. \tag{16}$$

4. *The Regions of Positive and Negative Density Gradients in the $x-y$ Plane.*—Equation (14) is our fundamental differential equation, and this equation defines in the $x-y$ plane a one-parametric family of curves. We will now show that the $x-y$ plane can be divided into two regions such that if the appropriate solution curve of (14) lies in one of the regions then we have a normal positive density gradient, while if the appropriate solution curve crossed over into the other region then we would have a negative density gradient.

From (9) we have that

$$\frac{dy}{y}=\frac{d\rho}{\rho}-3\frac{dT}{T}, \tag{17}$$

or

$$\frac{1}{\rho}\frac{d\rho}{dx}=\frac{1}{y}\frac{dy}{dx}+\frac{3}{T}\frac{dT}{dx}. \tag{18}$$

By (16),

$$\frac{dx}{x}=-4\gamma\frac{dT}{T}. \tag{19}$$

Hence

$$\frac{1}{\rho}\frac{d\rho}{dx}=\frac{1}{y}\frac{dy}{dx}-\frac{3}{4\gamma}\frac{1}{x}, \tag{20}$$

or using (14),

$$\frac{1}{\rho}\frac{d\rho}{dx}=\frac{y(1+\frac{1}{4}y)-x}{\gamma xy^2}. \tag{21}$$

(21) is easily seen to be equivalent to

$$\frac{1}{4}\left(\frac{T}{x}\right)\frac{1}{\rho}\frac{d\rho}{dT}=\frac{x-y(1+\frac{1}{4}y)}{xy^2}. \tag{22}$$

Now the temperature *always* increases inwards, so that we have a positive or a negative density gradient according as $d\rho/dT$ is positive or negative, *i.e.* according as

$$x\gtrless y(1+\tfrac{1}{4}y). \tag{23}$$

From (23) it follows that the *parabola*

$$x = y(1 + \tfrac{1}{4}y) \qquad (24)$$

divides the $x-y$ plane in such a way that for all points below the curve (24) *we have positive density gradients, while for all points above* (24) *we have negative density gradients.*

In fig. 1 we have drawn this critical parabola. We shall refer to the

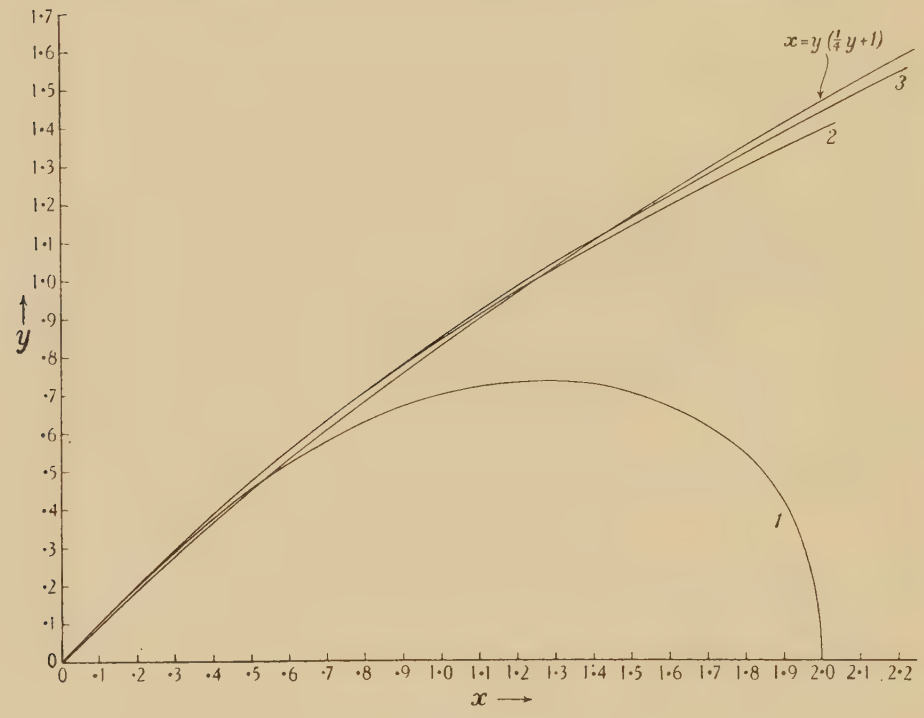

FIG. 1.

$$1 \to \frac{1}{x} = \tfrac{1}{2} - z - \log(1-z)$$

$$2 \to \frac{1}{x} = -z - \log(1-z) \qquad \textit{(Standard Solution)}$$

$$3 \to \frac{1}{x} = -\tfrac{1}{20} - z - \log(1-z)$$

The region below the parabola $x = y(\frac{1}{4}y+1)$ defines the positive region $\left(\frac{d\rho}{dT} > 0\right)$ *and the region above the* negative region $\left(\frac{d\rho}{dT} < 0\right)$.

region below (24) as the *positive region*, and the region above (24) as the *negative region.*

It follows, then, that to determine the conditions under which we can have negative density gradients, we have simply to determine the conditions under which a solution curve of (14) crosses (or lies entirely above) the curve (24).

5. *Transformations of the Differential Equation* (14).—For a discussion of (14) it is convenient to transform it into other equivalent forms.

First introduce the variable q defined by

$$q = x^{\frac{1}{\gamma}} \quad \text{or} \quad x = q^{\gamma}. \tag{25}$$

Equation (14) now takes the form

$$yq\frac{dy}{dq} = y(y+1) - q^{\gamma}. \tag{26}$$

Introduce now the new variable z defined by

$$y = zq. \tag{27}$$

(26) now transforms into

$$zq^{3}\frac{dz}{dq} = zq - q^{\gamma}. \tag{28}$$

The variables z and q can be expressed in terms of the physical variables ρ and T as follows :—

$$q = x^{\frac{1}{\gamma}} = K^{\gamma-1}T^{-4}. \tag{29}$$

Further, it is found that

$$z = yq^{-1} = \frac{k}{\mu H}\frac{3}{a}\frac{1}{K^{\gamma-1}}\frac{\rho}{T}. \tag{30}$$

From (29) and (30) we get

$$p = \frac{k}{\mu H}\rho T = \tfrac{1}{3}aK^{\gamma-1}z, \tag{31}$$

$$p_r = \tfrac{1}{3}aT^4 = \tfrac{1}{3}aK^{\gamma-1}\frac{1}{q}, \tag{32}$$

$$\rho = \frac{a}{3}\frac{\mu H}{k}K^{\gamma-1}zT^{-1} = \tfrac{1}{3}a\frac{\mu H}{k}K^{\frac{1}{4}\gamma-1}zq^{\frac{1}{4}}, \tag{33}$$

$$P = p + p_r = \tfrac{1}{3}aK^{\gamma-1}\left(z + \frac{1}{q}\right). \tag{34}$$

Sometimes it is more convenient to use the reciprocal of q as the variable. Write

$$q = t^{-1}. \tag{35}$$

(28) transforms into

$$z\frac{dz}{dt} = \frac{1}{t^{\gamma-1}} - z. \tag{36}$$

6. *The Differential Equation for the Density Distribution.*—So far we have not considered explicitly the equation

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho, \tag{37}$$

or its equivalent

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{38}$$

From (34) we have

$$\frac{dP}{dr} = \tfrac{1}{3}aK^{\gamma-1}\frac{d}{dr}\left(z + \frac{1}{q}\right). \tag{39}$$

Now
$$\frac{d}{dr}\left(z+\frac{1}{q}\right)=\left(\frac{dz}{dq}-\frac{1}{q^2}\right)\frac{dq}{dr}; \tag{40}$$

or using (28) we have finally that

$$\frac{dP}{dr}=-\tfrac{1}{3}aK^{\gamma-1}\frac{q^{\gamma-3}}{z}\frac{dq}{dr}. \tag{41}$$

Using (33) and (41) in (38) we have

$$\left\{\frac{1}{4\pi G}\frac{3}{a}\left(\frac{k}{\mu H}\right)^2 K^{-\frac{1}{2}\gamma-1}\right\}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{z^2}q^{\gamma-3\frac{1}{4}}\frac{dq}{dr}\right)=zq^{\frac{1}{4}}. \tag{42}$$

Put

$$r=\left[\frac{1}{4\pi G}\left(\frac{k}{\mu H}\right)^2\frac{3}{a}K^{-\frac{1}{2}\gamma-1}\right]^{\frac{1}{2}}\xi. \tag{43}$$

We have

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{z^2}q^{\gamma-3\frac{1}{4}}\frac{dq}{d\xi}\right)=zq^{\frac{1}{4}}. \tag{44}$$

Since from (27)

$$\frac{dq}{dz}=\frac{zq^3}{zq-q^\gamma},$$

we can also write (44) in the form

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\frac{\xi^2}{zq^{\frac{1}{4}}(zq^{1-\gamma}-1)}\frac{dz}{d\xi}\right)=zq^{\frac{1}{4}}. \tag{45}$$

7. *The Integration of the Special Case* $\gamma=1$.—From (28) we see that the case $\gamma=1$ integrates out immediately. For this case

$$q=x=KT^{-4}, \tag{46}$$

i.e. x is a measure of the reciprocal of the radiation pressure p_r. We can rewrite (28) as

$$zx^3\frac{dz}{dx}=x(z-1), \tag{47}$$

or

$$\frac{z}{z-1}dz=\frac{dx}{x^2}, \tag{48}$$

from which we immediately have

$$\frac{1}{x}=C-\{z+\log(|z-1|)\}, \tag{49}$$

where C is a constant of integration. We have, further, that

$$y=zx. \tag{50}$$

(49) and (50) together define parametrically the solution curves in the $x-y$ plane.

8. *Discussion of the Solution* (*Case* $\gamma=1$).—For purposes of discussion it is convenient to consider the two cases $z>1$ and $z<1$ separately.

Case I ($z > 1$).—We shall write (49) and (50) in the form

$$\frac{1}{x} = A - z - \log(z-1); \qquad y = zx. \qquad (z > 1) \tag{51}$$

From (51) we see that as

$$z \to 1+0, \qquad x \to 0, \qquad y/x \to 1, \tag{52}$$

i.e. the solution curves (51) touch the line $x=y$ at the origin, and since, further, $z > 1$, all these solution curves lie entirely above the line $x=y$. Hence they are always in the *negative region* in the $x-y$ plane, and correspond therefore to negative density gradients.

Further, for a given value of A (positive or negative) we can find a value for z, say z_0, such that

$$A = z_0 + \log(z_0 - 1). \qquad (z_0 > 1) \tag{53}$$*

Then from (51) it follows that as

$$z \to z_0, \qquad x \to \infty, \qquad y \to \infty, \qquad y/x \to z_0, \tag{54}$$

i.e. the solution curves tend asymptotically (from above) to the linear solutions $y = z_0 x$. In particular the solution curves approach the line $x=0$, as $A \to +\infty$. Again, as $A \to -\infty$, the solutions tend the line $x=y$. In fact, from (14) we see that $x=y$ *is also a solution of the equation. This last "singular" solution corresponds to a density variation inversely as the temperature.* From (31) we have

$$\frac{k}{\mu H}\rho T = \tfrac{1}{3}aK. \tag{55}$$

Case II ($z < 1$).—We shall now write (49) and (50) as

$$\frac{1}{x} = B - z - \log(1-z); \qquad y = zx. \tag{56}$$

Three sub-cases have to be considered :

Case II (a), ($B > 0$).—From (56) we see that as

$$z \to 0, \qquad x \to \frac{1}{B}, \qquad y \to 0. \tag{57}$$

Hence in this case the solution curves intersect the x-axis at a finite point. From (14) we verify that the *solution curves intersect the x-axis vertically.* (We are not concerned with the behaviour of the solution curves when x or y (or both) take negative values.) Since the solution (56) passes through the origin (touching the line $x=y$ at the origin), it must have a maximum somewhere between $z=0$ and $x=B^{-1}$. The locus of the maxima of these solutions ($B > 0$) is the parabola

$$x = y(y+1), \tag{58}$$

which lies entirely *below* the critical parabola

$$x = y(\tfrac{1}{4}y+1). \tag{59}$$

* Thus if $A=2$, $z_0=2$.

We see then that the solution curves (56) ($B > 0$) do enter the *positive region* in the $x-y$ plane. One easily verifies that near the origin these solution curves lie above the parabola (59), and hence are initially in the negative region (see fig. 1). Hence for a given B there exists a value for x, say $x(B)$, such that if $x > x(B)$ the density gradient in the configuration is positive and negative otherwise. In fig. 2 we have illustrated the run of the solutions for definite values of B.

Case II (*b*) ($B=0$).—From (57) we see that as $B \to 0$, the point where the solution curve (56) ($B > 0$) intersects the x-axis tends to infinity. In the limit $B=0$, the corresponding solution curve

$$\frac{1}{x} = -z - \log(1-z); \qquad y = zx, \tag{60}$$

does not intersect the x-axis for any finite x. From (60) we have that

$$\frac{1}{x} = -z + (z + \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 + \ldots). \tag{60'}$$

Hence

$$\frac{1}{x} = \tfrac{1}{2}z^2(1 + \tfrac{2}{3}z) + O(z^4). \tag{61}$$

From (61) we see that as $z \to 0$, $x \to \infty$:

$$x \sim \frac{2}{z^2} = \frac{2x^2}{y^2}, \tag{62}$$

or

$$y^2 \sim 2x. \qquad (x, y \to \infty) \tag{63}$$

From (33) and (34) we have that as $z \to 0$,

$$P \sim \tfrac{1}{3}aKz \sim \tfrac{1}{3}aK\sqrt{\frac{2}{x}}, \tag{64}$$

$$\rho = \tfrac{1}{3}a\frac{\mu H}{k}K^{3/4}zx^{1/4} \sim \tfrac{1}{3}a\frac{\mu H}{k}K^{3/4}\frac{\sqrt{2}}{x^{1/4}}, \tag{65}$$

or

$$P = D\rho^2, \qquad (y \to \infty) \tag{66}$$

where

$$D = \frac{3}{a}\left(\frac{k}{\mu H}\right)^2 \frac{1}{\sqrt{2K}}. \tag{67}$$

From (63) it follows that as $x \to \infty$, the solution curve (60) is in the positive region. However, near the origin the curve lies above the parabola (59), and it is found to cross it for $y = 0{\cdot}99$ (approximately) (see fig. 1). It follows, then, that *if the central radiation pressure* ($1-\beta_c$) *is greater than a half* (*approximately*), *then there occur negative density gradients, while if* ($1-\beta_c$) $< 0{\cdot}5$ *we have only positive density gradients.* We shall see presently (§ 10) that (59) is physically a very important case, and consequently the above remark in italics (so far as it goes) justifies the conjecture made in § 1. We shall refer to (59) as the *standard solution.* This solution is tabulated in Table I.

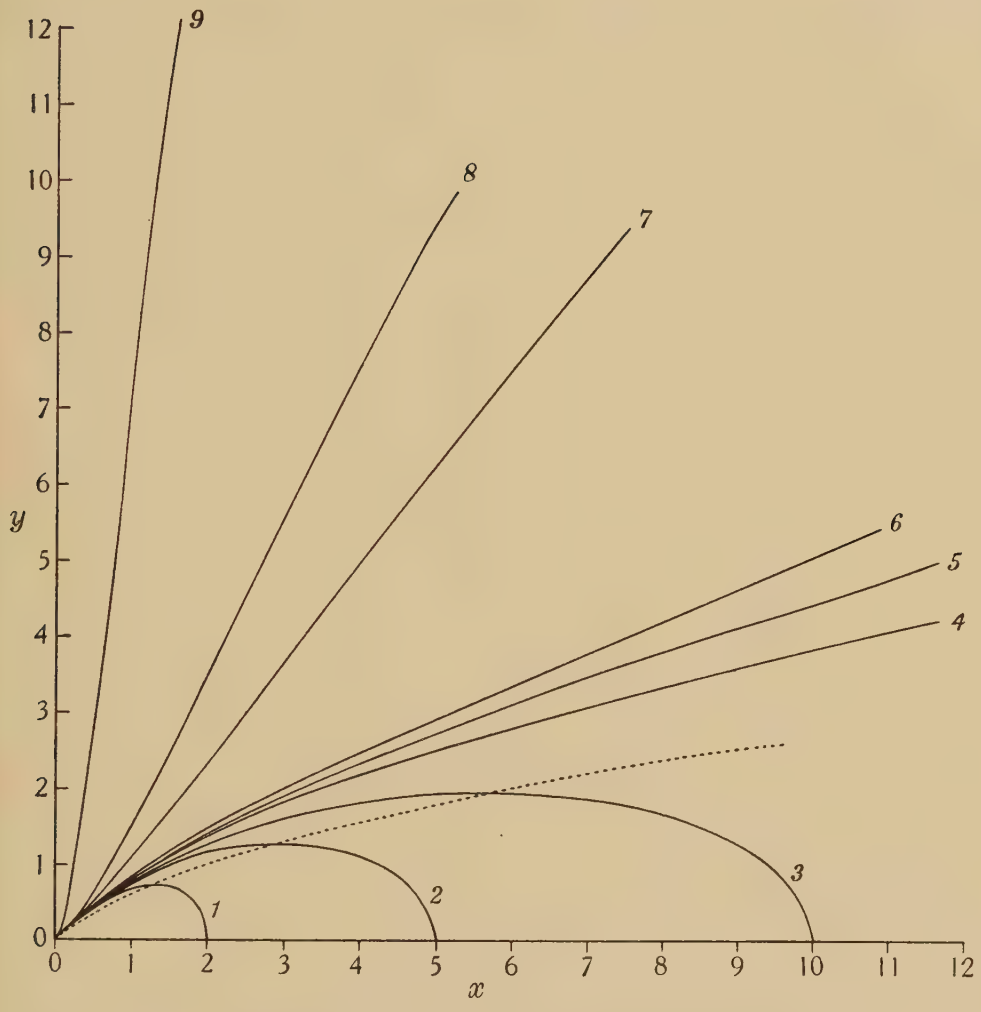

FIG. 2.—*The solution curves of the differential equation* (14) *for* $\gamma = 1$. *The explicit solution is*

$$\left.\begin{aligned} \frac{1}{x} &= A - z - \log(z-1) \\ \text{or} \qquad & \\ \frac{1}{x} &= B - z - \log(1-z) \end{aligned}\right\} y = zx$$

A and B being constants

$1 \to B = \frac{1}{2}$;	$4 \to B = 0$;	$7 \to A = 0$
$2 \to B = \frac{1}{5}$;	$5 \to B = -\frac{1}{20}$;	$8 \to A = 2$
$3 \to B = \frac{1}{10}$;	$6 \to B = -\frac{1}{10}$;	$9 \to A = 10$

Curve 4 *is the standard solution and the dotted curve is the parabola* $x = y(y+1)$ *which is the locus of the maxima of the solutions of the type* 1, 2 *or* 3.

TABLE I

The Standard Solution $\frac{1}{x} = -z - \log(1-z)$

z	x	y	σ	τ
0·999	0·172	0·172	0·644	1·553
·995	·232	·231	·691	1·440
·990	·277	·274	·718	1·379
·980	·341	·334	·749	1·309
·970	·391	·382	·769	1·262
·950	·489	·464	·794	1·196
·925	·601	·555	·814	1·136
·900	·713	·642	·827	1·088
·875	·830	·726	·835	1·048
·850	·955	·812	·840	1·012
·825	1·089	·899	·843	·979
·800	1·235	·988	·843	·949
·750	1·572	1·179	·840	·893
·700	1·984	1·389	·831	·843
·650	2·501	1·626	·817	·795
·600	3·162	1·897	·800	·750
·550	4·024	2·213	·779	·706
·500	5·178	2·589	·754	·663
·400	9·023	2·609	·693	·577
·300	17·65	5·294	·615	·488
·200	43·2	8·647	·513	·390
·100	186·2	18·62	·369	·271
0	∞	∞	0	0

Case II (*c*) ($B < 0$).—If B were negative, then for some value of z, say $z = z_0 (< 1)$, we have

$$B = z_0 + \log(1 - z_0). \qquad (z_0 < 1) \qquad (68)$$

From (68) it follows that as $z \to z_0$, the solution becomes asymptotic to the line $y = z_0 x$. (All these solutions, of course, lie entirely below the line $x = y$.) Since all these solutions tend to infinity linearly, it follows that all of them eventually lie in the negative region. Further, *all* the solutions lie *initially* above the critical parabola and hence are in the negative region.

Now if B were negative and very nearly zero, the solution curves then cross into the positive region before again getting out into the negative region. Hence for such solutions there exists a finite interval, say $x_1 \leqslant x \leqslant x_2$,* corresponding to which we have positive density gradients. For $x < x_1$ or $x < x_2$ we have negative density gradients. It is clear, then, that there exists a negative value for B, say B^*, such that the corresponding solution curve touches the critical parabola, and for all negative values of B such that $-B > -B^*$, the solution curves lie entirely in the negative region. The situation is clear on an examination of figs. 1 and 2.

* x_1 and x_2 of course depend upon B.

9. *Discussion of the Solution* ($\gamma = 1$) *in the* (ρ, T) *Plane.*—To translate the results of § 8 explicitly in terms of the possible types of density-

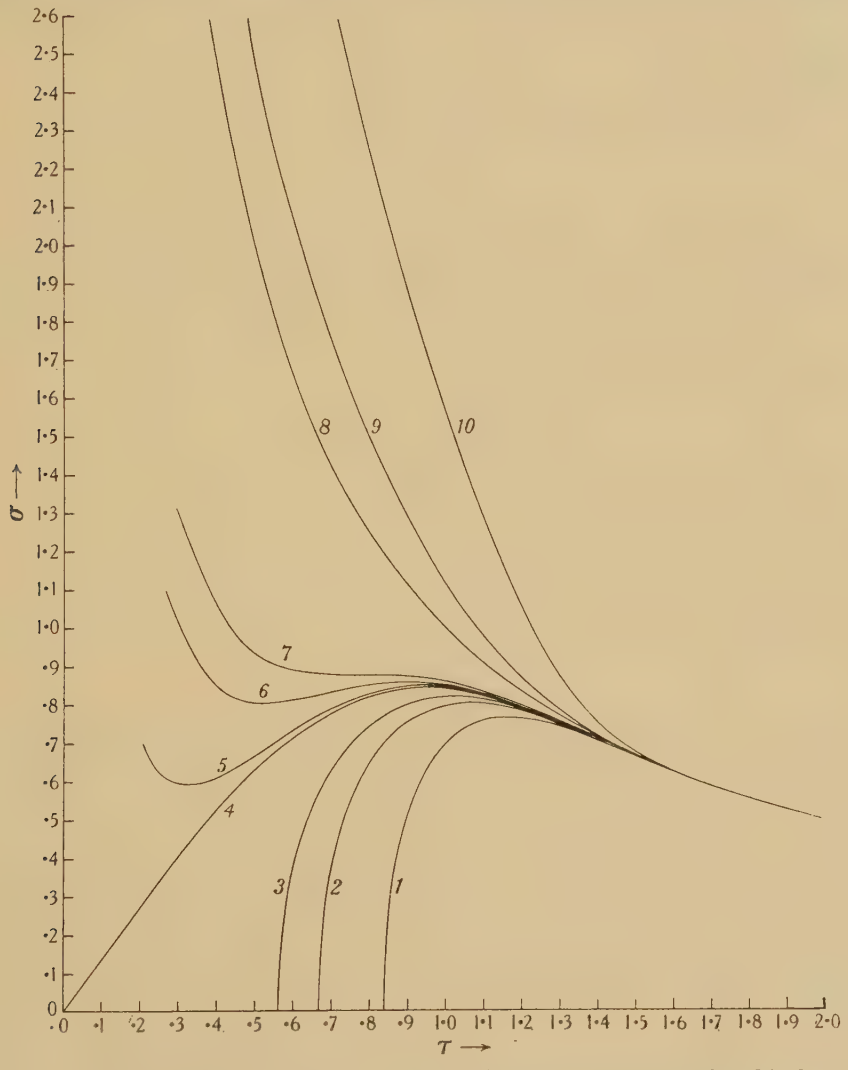

FIG. 3.—*The density* (σ)*-temperature* (τ) *relations for the solution curves plotted in fig.* 2.

$1 \to B = \frac{1}{2}$; $4 \to B = 0$; $7 \to B = -\frac{1}{10}$
$2 \to B = \frac{1}{5}$; $5 \to B = -\frac{1}{100}$; $8 \to \sigma\tau = 1$
$3 \to B = \frac{1}{10}$; $6 \to B = -\frac{1}{20}$; $9 \to A = 0$; $10 \to A = 2$.

temperature variations, we shall first rewrite the equations (32) and (34) ($\gamma = 1$) as follows :—

$$\rho = \rho_0 \sigma ; \qquad T = T_0 \tau, \tag{69}$$

where

$$\rho_0 = \tfrac{1}{3} a \cdot \frac{\mu H}{k} \cdot K^{3/4} ; \qquad T_0 = K^{1/4}, \tag{70}$$

$$\sigma = z x^{1/4} ; \qquad \tau = x^{-1/4}, \tag{71}$$

σ and τ are then the density and the temperature measured in suitable scales. In Tables II and III we have tabulated σ and τ (with z as argument) corresponding to the different types of solutions considered in § 8 with definite numerical values for the integration constant. The results are graphically illustrated in fig. 3.

The results can be summarized as follows :—

(i) All the solutions are asymptotic to the rectangular hyperbola

$$\sigma\tau = 1, \tag{72}$$

as $\tau \to \infty$. The above is simply the restatement of our earlier result that in the $(x-y)$ plane all the solutions touch the line $x=y$ at the origin.

(ii) The solution curves above the hyperbola $\sigma\tau = 1$ are all characterized by negative density gradients $\dfrac{d\rho}{dT} < 0$.

TABLE II

The Density-temperature Relations corresponding to the Solutions $\dfrac{1}{x} = B - z - \log(1-z)$

z	$B=\frac{1}{2}$		$B=\frac{1}{5}$		$B=\frac{1}{10}$		$B=-\frac{1}{100}$		$B=-\frac{1}{20}$		$B=-\frac{1}{10}$	
	σ	τ	σ	τ	σ	τ	σ	τ	σ	τ	σ	τ
0·999	0·630	1·585	0·638	1·566	0·641	1·559	0·644	1·552	0·645	1·549	0·646	1·546
·995	·672	1·480	·683	1·457	·687	1·449	·691	1·440	·693	1·436	·695	1·432
·990	·695	1·424	·708	1·398	·713	1·388	·718	1·378	·720	1·374	·723	1·369
·980	·720	1·361	·737	1·330	·743	1·320	·750	1·307	·752	1·303	·755	1·297
·970	·740	1·310	·754	1·286	·761	1·274	·769	1·261	·772	1·256	·776	1·250
·950	·752	1·263	·776	1·224	·785	1·210	·795	1·195	·799	1·189	·804	1·181
·925	·763	1·213	·792	1·169	·802	1·153	·816	1·134	·821	1·127	·827	1·119
·900	·766	1·174	·800	1·125	·813	1·107	·829	1·086	·835	1·078	·842	1·068
·875	·766	1·143	·804	1·089	·819	1·069	·837	1·045	·844	1·037	·854	1·025
·850	·762	1·115	·804	1·057	·821	1·035	·842	1·009	·851	·999	·862	·987
·825	·756	1·091	·802	1·028	·821	1·004	·845	·976	·855	·965	·867	·951
·800	·748	1·070	·798	1·002	·819	·977	·846	·946	·857	·934	·871	·918
·750	·726	1·032	·784	·956	·810	·926	·843	·890	·859	·873	·877	·856
·700	·699	1·001	·764	·916	·794	·882	·835	·838	·853	·821	·878	·797
·650	·667	·974	·739	·880	·773	·841	·823	·790	·845	·769	·878	·740
·600	·631	·951	·708	·848	·747	·803	·807	·744	·835	·718	·880	·682
·550	·591	·930	·672	·818	·716	·768	·787	·699	·824	·668	·886	·621
·500	·548	·913	·632	·792	·680	·736	·764	·654	·813	·615	·905	·552
·450	...	...	...	...	...	...	...	...	...	...	·962	·468
·400	·452	·884	·536	·747	·590	·678	·710	·563	·805	·497	1·240	·323
·350	...	...	...	...	...	...	...	...	·834	·419		
·300	·347	·864	·422	·712	·477	·629	·646	·465	1·050	·286		
·200	·235	·851	·291	·687	·338	·592	·591	·339				
·100	·119	·843	·149	·673	·176	·570	·597	·293				
0	0	·841	0	·669	0	·562	·669	·224				

TABLE III

The Density-temperature Relations corresponding to the Solutions $\frac{1}{x} = A - z - \log(z-1)$

$A=10$			$A=2$			$A=0$		
z	σ	τ	z	σ	τ	z	σ	τ
1	∞	∞	1·025	·698	1·470	1·005	·637	1·578
2	1·189	1·682	1·05	·745	1·409	1·010	·733	1·379
3	1·893	1·585	1·1	·822	1·338	1·02	·782	1·304
4	2·688	1·488	1·2	·963	1·246	1·04	·856	1·215
5	3·626	1·379	1·3	1·107	1·175	1·06	·921	1·151
6	4·825	1·244	1·4	1·262	1·110	1·08	·985	1·097
7	6·677	1·048	1·5	1·435	1·045	1·10	1·050	1·047
7·5	8·425	·890	1·6	1·638	0·977	1·25	1·237	·930
8	16·6	·482	1·7	1·889	·900	1·20	1·500	·799
			1·8	2·232	·807	1·25	2·057	·608
			1·9	2·822	·673	1·275	3·584	·356
			2·0	∞	0			

(iii) Inside the hyperbola (72) there is another "critical" curve which passes through the origin. Near the origin the solution is linear and we have

$$\sigma = \sqrt{2}\,\tau. \qquad (\tau \to 0) \tag{73}$$

The solution here considered corresponds to Case II (*b*) of § 8, *i.e.*

$$\frac{1}{x} = -z - \log(1-z); \qquad y = zx. \tag{74}$$

The result (73) is easily obtained from (62). We shall refer to the curve in the (σ, τ) plane which corresponds to (74) as I.

(iv) In the region bounded by $\sigma\tau = 1$ and the curve I (*cf.* (iii) above) there is just one solution which is characterized by $\frac{d^2\sigma}{d\tau^2} = 0$, for a definite finite value of τ. Denote this curve by II.

(v) Any solution between I and II has both a maximum and a minimum, and there exists only a definite interval in τ within which the density gradient is positive.

(vi) All solutions between the hyperbola $\sigma\tau = 1$ and the curve II correspond to having negative density gradients for all values of τ.

(vii) All solutions below I and the τ-axis start from the τ-axis at a finite τ, reach a maximum and decrease again, becoming asymptotic to the hyperbola $\sigma\tau = 1$ as $\tau \to \infty$.

On comparing our fig. 3 with the figure in Rosseland's paper, we see that our model reproduces all the different types of density distributions that are possible. We have in addition shown the *theoretical* possibility of negative

density gradients for all values of the temperature. Further, our analysis gives insight into the situation and shows precisely how the phenomenon of the negative density gradient does arise.

10. In § 8 we have seen with reference to one particular case ($\gamma = 1$) how the solutions of the differential equation (14) define a one-parametric family of curves in the $x-y$ plane. The solution curves in the case we have discussed were labelled by the numerical values of the integration constant A or B according as the solution curve lies entirely above or below the line $x=y$ ($x=y$ is a *singular solution* of the equation). The particular solution curve which has to be selected for a given equilibrium configuration (*i.e.* prescribed L and M) will naturally depend upon the boundary condition that is to be adopted. Thus if, for instance, we assumed that the law $\eta \propto T^{4\frac{1}{2}}$ was valid (formally) for the "whole" configuration, then the usual boundary condition, namely, that $y=0$ at $T=0$, or what is the same thing, $y=0$ at $x=\infty$, leads us to select our "*standard solution*"

$$\frac{1}{x} = -z - \log(1-z); \qquad y = zx,$$

as the appropriate solution for the model under consideration.

There is, however, a more important case where the interior is described by our standard solution. This is the case where we consider the star as consisting of two regions, the inner region in which the law $\eta \propto T^{4\frac{1}{2}}$ is valid, and an outer envelope with a uniform source distribution. If T^* is the temperature at the interface, then the corresponding value x^* of x is according to (5) and (16) easily verified to be

$$x^* = \frac{4\pi cGM}{\kappa_1 L}\frac{3}{a}\frac{k}{\mu H}T^{*\frac{1}{2}}. \tag{75}$$

Using Strömgren's value for κ_1, namely, that

$$\kappa_1 = 3{\cdot}89 \times 10^{25}(1-X^2), \tag{76}$$

where X is the abundance of hydrogen by weight, we find that (75) reduces to

$$x^* = 1{\cdot}1 \times 10^4 \frac{M}{L\mu(1-x^2)}\left(\frac{T^*}{10^6}\right)^{1/2}, \tag{77}$$

where M and L are now expressed in solar units.

Now in the envelope the equation for y is exactly the same as has been studied in a recent paper by the author.* From equations (21), (23) and (37) of the paper just quoted we verify that

$$y^*(y^*+1) = \frac{4\pi cGM}{\kappa_1 L}\frac{3}{a}\frac{k}{\mu H}T^{*\frac{1}{2}}, \tag{78}$$

where y^* is the value of y at the interface. Combining (75) and (78) we have that

$$x^* = y^*(y^*+1). \tag{79}$$

* S. Chandrasekhar, *M.N.*, 96, 647, 1936. What is denoted by λ in this paper is our present y.

(79) shows that our solution curve must intersect the parabola $x=y(y+1)$, from which we conclude that the solution must be of the type discussed in § 8 under Case II (*a*). Further, from (77) we see that, *normally*, the value, of x at which the intersection with the parabola $x=y(y+1)$ occurs is of the order of 10^4. Hence the appropriate solution curve must intersect the x-axis at a very large value for x, i.e. B should be very nearly zero. From this it follows that for the model under discussion the inner regions (except near the interface) will, very approximately, be described by our standard solution. We are thus in a position to appreciate the importance of the standard solution. Incidentally we have been led to consider "composite" stellar models with different laws for the "energy generation" in the two different regions—only a beginning has been made in this section, but we shall consider this problem in greater detail in our second paper of this series.

11. *Models with* $\eta \propto \rho^\alpha T^\delta$.—We shall now generalize the analysis of § 3 to the case where $\eta \propto \rho^\alpha T^\delta$. Let

$$\eta=\eta_0\rho^\alpha T^\delta. \tag{80}$$

By an analysis similar to that of § 3 we get that

$$p_r\frac{dy}{dp_r}=K\frac{T^{(\frac{1}{2}-\delta-3\alpha)}}{y^{\alpha+1}}-(1+y), \tag{81}$$

where y has the same meaning as before, and

$$K=\frac{4\pi cGM}{\kappa_1\eta_0 L}\left[\frac{3}{a}\frac{k}{\mu H}\right]^{\alpha+1}. \tag{82}$$

On introducing the new variable x defined by

$$x=KT^{(\frac{1}{2}-\delta-3\alpha)}, \tag{83}$$

(81) transforms into

$$\gamma x y^{\alpha+1}\frac{dy}{dx}=y^{\alpha+1}(y+1)-x, \tag{84}$$

where

$$\gamma=\tfrac{1}{4}(\delta+3\alpha-\tfrac{1}{2}). \tag{85}$$

Equation (84) now corresponds to our earlier equation (14). If we now introduce the new variables defined by

$$q^\nu=x\,; \qquad y=zq, \tag{86}$$

(84) goes over into

$$z^\nu q^{\nu+2}\frac{dz}{dq}=z^\nu q^\nu-q^\gamma. \qquad (\nu=\alpha+1) \tag{87}$$

12. *Integration of the Case* $\gamma=\nu$.—From (87) we see that the case $\gamma=\nu$ can be explicitly integrated for specified values of $\gamma=\nu$. In this case

$$\gamma=\tfrac{1}{4}(\delta+3\alpha-\tfrac{1}{2})=\nu=\alpha+1, \tag{88}$$

or

$$\delta=\alpha+4\tfrac{1}{2}. \tag{89}$$

Our law for the variation of η is therefore

$$\eta = \eta_0 \rho^{\alpha} T^{\alpha+4\frac{1}{2}}. \tag{90}$$

When $\alpha = 0$, (90) corresponds to the case we have already investigated.

When $\gamma = \nu$, (87) can be rewritten as

$$\frac{dq}{q^2} = \frac{z^{\nu}}{z^{\nu} - 1} dz, \tag{91}$$

or

$$\frac{1}{q} = C - z - \int \frac{dz}{z^{\alpha+1} - 1}, \tag{92}$$

where C is the constant of integration. When α is given a specified numerical value the integral in (92) can be explicitly evaluated. Thus when $\alpha = -\frac{1}{2}$, 1 or 2, we have respectively

$$\frac{1}{q} = C - \{z + 2z^{1/2} + 2 \log |(1 - z^{1/2})|\}. \qquad (\alpha = -\tfrac{1}{2}) \tag{93}$$

$$\frac{1}{q} = C - \left\{z + \log\left(\left|\frac{z-1}{z+1}\right|\right)^{1/2}\right\}. \qquad (\alpha = 1) \tag{94}$$

$$\frac{1}{q} = C - \left\{z + \tfrac{1}{6} \log \frac{(z-1)^2}{z^2+z+1} - \frac{1}{\sqrt{3}} \tan^{-1} \frac{2z+1}{\sqrt{3}}\right\}. \qquad (\alpha = 2) \tag{94'}$$

The solutions (92), (93) and (94) can be discussed by the methods of §§ 8 and 9, but we shall reserve these for a future communication. We shall, however, obtain here the results for the solution (92) analogous to those obtained in § 8, Case II (*b*)—*i.e.* the "*standard solution*" for our present model.

From (92) we easily verify that the limiting solution which intersects the x-axis at infinity can be written as

$$\frac{1}{q} = \int_0^z \frac{dz}{1 - z^{\alpha+1}} - z. \tag{95}$$

When $z \to 0$, we have

$$\frac{1}{q} = \frac{1}{\alpha+2} z^{\alpha+2} + \frac{1}{2\alpha+3} z^{2\alpha+3} + \frac{1}{3\alpha+4} z^{3\alpha+4} + \ldots \tag{96}$$

or

$$z \sim (\alpha+2)^{\frac{1}{\alpha+2}} x^{-\frac{1}{(\alpha+2)(\alpha+1)}}. \qquad (z \to 0,\ x \to \infty) \tag{97}$$

Now as $x, y \to \infty$, we have

$$P \sim \frac{a}{3} K^{\frac{1}{\alpha+1}} z, \tag{98}$$

$$\rho = \tfrac{1}{3} a \frac{\mu H}{k} K^{\frac{3}{4(\alpha+1)}} z q^{1/4}, \tag{99}$$

$$\sim \frac{(\alpha+2)^{1/4}}{3} a \left(\frac{\mu H}{k}\right) K^{\frac{3}{4(\alpha+1)}} z^{\frac{2-\alpha}{4}}, \tag{100}$$

or

$$P = D\rho^{\frac{4}{2-\alpha}}, \tag{101}$$

where

$$D = \left[\left(\frac{3}{a}\right)^{2+\alpha}\left(\frac{k}{\mu H}\right)^{4}\frac{1}{(\alpha+2)K}\right]^{\frac{1}{2-\alpha}}. \tag{102}$$

Hence configurations built on this model and which have negligible radiation pressure (y large) are polytropic distributions with an index n related to α by the relation

$$n = \frac{2-\alpha}{2+\alpha} \tag{103}$$

—in particular when $\alpha = -1$, $n = 3$, as we should have expected.

(103) brings out explicitly the critical nature of $\alpha = 2$ for these models. From (100) (and also (103)) one infers that for $\alpha = 2$ the corresponding standard solution predicts *uniform density* for $T \to 0$:

$$\rho \to \frac{\sqrt{2}}{3}\alpha\frac{\mu H}{k}K^{\frac{1}{4}} = \text{constant.} \qquad (T \to 0) \tag{104}$$

When $\alpha > 2$ we have negative density gradients for *all* values of T even for the standard solution. The nature of the (σ, τ) relations can be understood by discussing the case $\alpha = 3$ when

$$\frac{1}{q} = C - \left\{z + \tfrac{1}{4}\log\left(\left|\frac{z-1}{z+1}\right|\right) - \tfrac{1}{2}\tan^{-1}z\right\}. \tag{105}$$

These models will be considered in greater detail in our second paper, which will also include a discussion of the *stability* of these models.

So far we have only discussed the different types of density distributions that are formally possible by concentrating the energy sources sufficiently towards the centre. We have seen the fundamental rôle which the *magnitude* of the radiation pressure ($= 1 - \beta$) plays in the occurrence or otherwise of negative density gradients. The results have a physical significance in view of Rosseland's suggestion that a state of frequent collapsing of matter towards the centre of a star is a *possibility* when negative density gradients occur. Further, our study indicates the importance of this phenomenon, especially for the massive stars—we have probably here an explanation for the "homogeneity" of the Trumpler stars disclosed in an earlier study of the author.* It may be mentioned here that in the writer's view the whole question of "convection" in the interior of stars will have to be discussed from a physical point of view quite different from that which has been generally adopted. But these and other related matters will be the subject of a future communication.

Trinity College, Cambridge:
1936 *December* 5.

* *M.N.*, 96, 647, 1936.

ON THE GALACTIC ROTATION OF THE GLOBULAR CLUSTERS SYSTEM: SOME COMMENTS ON F. K. EDMONDSON'S PAPER.

Dr. Henri Mineur.

1. I recently published * a paper on the rotation of the globular clusters which led me to conclude that these celestial objects, as a whole, were revolving around the axis of the Milky Way with a velocity which can be approximately represented by the expression

$$V = -550r + 240r^2 + 280rz^2 \quad \text{km./sec.},$$

r being the distance of the cluster from the axis of the galaxy and z its distance from the galactic plane. (Unit: 10^4 parsecs.) At the same time Mr. Edmondson † published a work on the same problem. I feel it necessary to specify and to discuss the hypotheses we respectively adopted, Edmondson adopting in his work a standpoint very different from my own.

2. Edmondson assumes the motions of the clusters to be equally divided between direct and retrograde motions with regard to the axis of the Milky Way.

It should be pointed out that this hypothesis is fundamental in Edmondson's work. What Edmondson indeed calls absolute radial velocity is the velocity observed, corrected for the translation of the Sun only, this translation being determined with regard to the clusters as a whole. Thus determining the absolute radial velocities and taking them as rotational velocities, one must expect to find direct and retrograde velocities with zero as their mean value.

A rotation of the globular clusters system will simply express itself by a correlation between the absolute velocities and the position of the cluster in the galaxy. It does not appear that Edmondson attempted the analysis. Besides, we shall see later on that all the clusters are not adequate for such an investigation.

Edmondson assumes the clusters to describe circles having the same centre as the galaxy with a random distribution of the pole of the orbits. He admits indeed that the absolute spatial velocity of each cluster is perpendicular to the radius joining the cluster to the galactic centre. He does admit too, later on, that the magnitude of this velocity is given by a formula such as

$$V = kd^{-\frac{1}{2}},$$

where d is the distance of the cluster from the galactic centre, as is the case in the Keplerian circular motions.

Since we know nothing of the globular clusters but their positions and

* *M.N.*, 96, 61, 1936. † *A.J.*, 45, 1, 1935.

radial velocities, we cannot study them unless using hypotheses, the best being evidently to make as little of them as possible and not too arbitrary.

I find it difficult to agree with Edmondson's second hypothesis—orbits strictly circular with a velocity function of d only. Besides, I think it is imprudent to admit *a priori*, as he does, the clusters as a whole to be deprived of rotation.

3. Since the publication of our papers Edmondson has published a third memoir * where the two first ones are discussed. Here are the objections he raised against my work. I will try to answer them, taking the opportunity to make clearer my hypotheses and my reasonings.

Edmondson believes me to have made the three following hypotheses :—

(*a*) The orbit of the clusters are circular.
(*b*) They are parallel to the galactic plane, their centre being on the axis of the galaxy.
(*c*) The direction of the rotation is the same for all clusters.

Edmondson agrees with assumption (*a*), but rejects the assumptions (*b*) and (*c*).

I hasten to say I did not make either of those three hypotheses. Here is, on the contrary, the only assumption I used as a starting-point :

As a first approximation, the distribution of the velocities in the galaxy is of revolution with respect to the galactic axis.

I think this hypothesis is supported by what we know of the structure of the galaxy and of the extragalactic nebulæ.

It follows that if we consider a category of celestial bodies, the globular clusters, for instance, and among them those located in a small region of space, the mean of the velocities $\vec{v}$ of these later will be a vector $\vec{V}$ perpendicular to the galactic axis and whose magnitude will be a function of r and z only. The individual velocity $\vec{v}$ of a cluster will be

$$\vec{v} = \vec{V} + \vec{v_r}, \tag{1}$$

the mean of the vectors $\vec{v_r}$, the residual velocities, being zero. Also, ρ being the observed radial velocity and $\bar{\rho}$ the mean of the radial velocities of the clusters located in a small region, one has

$$\rho = \bar{\rho} + \rho_r + \epsilon_\rho, \tag{1'}$$

ρ_r being the radial component of v_r and ϵ_ρ, the error made in measuring ρ. $\bar{\rho}$ is the radial component of $\vec{V}$. Hence we have

$$\bar{\rho} = -\alpha X - \beta Y - \gamma Z + \frac{\beta x - \alpha y}{r} V, \tag{2'}$$

the notation being the same as in my previous work : α, β, γ = direction cosines of the direction sun-cluster ; x, y, z, co-ordinates of a cluster with regard to the galactic axes which have their centre o at the centre of the

* *M.N.*, 96, 636, 1936.

galaxy; X, Y, Z, components of the velocity of the Sun with regard to O. Z, distance of a cluster from the galactic axis.

The mean value of ϵ_ρ and ρ_r being zero, the coefficients of the right member of (2′) will be determined by applying a least squares solution to the equations obtained for each cluster by substituting ρ to $\bar{\rho}$ in formulæ (2′), that is to say, to equations (2) of my first paper :

$$\rho = -\alpha X - \beta Y - \gamma Z + \frac{\beta x - \alpha y}{r} V. \qquad (2)$$

$\vec{v_r}$ may have a radial component with regard to the galactic centre. So the orbits of the clusters are not supposed to be circular.

From the fact that $\vec{V}$ is perpendicular to the galactic axis, it does not follow that $\vec{v}$ is perpendicular to the same axis. Consequently I did not assume the orbits of the clusters to be circular and parallel to the galactic plane. Thus in a moving fluid the trajectories of molecules are distinct from the lines which are at all points tangent to the mean velocity of the molecules.

It may happen that vector $\vec{V}$ obtained through resolving equations (2) is always directed as in a retrograde motion, some clusters having nevertheless a direct motion. This will be the case if the residual velocity $\vec{v_r}$ of a cluster has a component opposite to $\vec{V}$ and greater in magnitude.

So I have not assumed that the direction of rotation is the same for all clusters.

Equations (2) may even give for V a function which keeps equal to zero. Within the limits of accuracy dealt with, the orbits of the clusters would now be indifferently direct or retrograde. But it is through the application of my method for studying radial velocities that such a result would eventually come to light.

4. Edmondson proposes to determine a by the condition that the velocity of the Sun with regard to O be precisely the vector V corresponding to $r = 1$, $z = 0$, basing his method on a remark I made that such is the case for the velocity of the Sun I have myself adopted. But I only mentioned this fact, as well as the agreement between the values of A and B and Oort's coefficients, as a curiosity. I think that not only would it be risky to determine a by the condition $\sqrt{X^2 + Y^2} = V$ (1, 0), but that it would go against the observed facts. It follows indeed from Strömberg's asymmetrical drift that the velocities of the various groups of celestial bodies which compose the Milky Way differ widely from one another.

5. Lastly, Edmondson judges that the formula I adopted to represent V,

$$V = ar + br^2 + crz^2, \qquad (3)$$

is lacking in flexibility, being limited to too few terms. I willingly agree, thinking myself that this formula is just fit to help us to get a rough idea of the rotation of the globular clusters if possible. This last objection incited

me to take up again my computations with different data and formulæ. I will develop elsewhere my new investigations.* The numerical results obtained by means of four new formulæ of interpolation fit in with those of my previous memoir. The globular clusters revolve with an important retrograde motion, which decreases with increasing distance from the galactic plane.

Summary

I analyse and discuss the fundamental hypotheses of the papers published by Mr. Edmondson and myself concerning the rotation of the globular clusters system.

I specially explain that I did not assume any of the hypotheses Edmondson charged me with ; I merely assumed the structure of the galaxy to be of revolution around the galactic axis.

* *Comptes rendus Acad. des Sciences de Paris*, **202**, 2, 132, 1936. See also *Bulletin Astronomique*, 1937.